6.1, 6.2 Introduction to Functions

We are familiar with function notation like $f(x) = x^3 - 3x^2 + x - 7$, which give us the rule of association. But there are a few other things to keep in mind.

Definition. Let A, B be sets. A *function* from set A to set B is a rule that associates to every element of the set A exactly one element of the set B.

We write $f : A \to B$ (f is a function from A to B).

A is called the *domain* of f, $A = \text{dom}(f)$.
B is called the *codomain* of f, $B = \text{codom}(f)$

If $a \in A$, $f(a)$ is called the *image* of a under f.
If $b \in B$, and $x \in A$ is such that $f(x) = b$, we say that x is a *preimage* of b under f.

The *range* of f is the set $\{f(x) \mid x \in A\}$, also called the image of the set A under f.

Note. $\text{range}(f) \subseteq \text{codom}(f)$, also $\text{range}(f) = \{y \in B \mid (\exists x \in A)(f(x) = y)\}$.

Example. We often use arrow diagrams to represent functions.
Let $A = \{1, 2, 3, 4\}$, $B = \{10, 11, 12, 13, 14\}$.

x	$f(x)$
1	
2	
3	
4	

y	set of preimages of y
10	
11	
12	
13	
14	

$\text{dom}(f) =$ $\qquad$ $\text{codom}(f) =$ $\qquad$ $\text{range}(f) =$

Example. Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^2 - 4x - 21$.

y	set of preimages of y
5	
0	
-25	
-30	

From graph: range$(f) =$

Note: $y_0 \in \text{range}(f) \iff$ the horizontal line $y = y_0$ intersects the graph
$\iff x^2 - 4x - 21 = y_0$ has a solution

With the second equivalence, we can find the range algebraically:

Example. The *number of divisors* function $d : \mathbf{N} \to \mathbf{N}$ is given by

$$d(n) = \text{the number of natural divisors of } n$$

n	divisors of n	$d(n)$
7		
12		
27		
30		

y	set of preimages of y
1	
2	
3	
4	
n	

range$(d) =$

Example. Let $\mathbf{Z}_4 = \{0, 1, 2, 3\}$, and define functions $f, g : \mathbf{Z}_4 \to \mathbf{Z}_4$.

$f(x) = x^2 \pmod 4$ $\quad\quad$ $g(x) = x^2 - 3x + 1 \pmod 4$

x	$f(x)$	$g(x)$
0		
1		
2		
3		

range(f) =

range(g) =

y	set of preimages of y under f	set of preimages of y under g
0		
1		
2		
3		

Example. Let $\mathbf{Z} \times \mathbf{Z} = \{(a, b) \mid a, b \in \mathbf{Z}\}$, where $(a_1, b_1) = (a_2, b_2)$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

Define $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ by $f(m, n) = 2m + n$. Determine and visualize the set of preimages of 3.

Definition. Two functions f and g are equal if:
1) $\text{dom}(f) = \text{dom}(g)$
2) $\text{codom}(f) = \text{codom}(g)$
3) $f(x) = g(x)$ for every $x \in \text{dom}(f)$.

Example. Functions $f, g : \mathbf{Z}_4 \to \mathbf{Z}_4$ from an earlier example are not equal.

Example. Are functions $f, g : \mathbf{R} \to \mathbf{R}$, $f(x) = x$, $g(x) = \sqrt{x^2}$ equal?

Example. Are functions $f(x) = \dfrac{x^2 - 4}{x - 2}$, $g(x) = x + 2$ equal?

This does not prevent us from writing $\dfrac{x^2 - 4}{x - 2} = x + 2$, because such equations are meant to be valid for all x for which both sides are defined.

6.3 Injections, Surjections, Bijections

We consider some basic properties of functions.

Definition. Let $f : A \to B$ be a function. We call f an *injection* (or a *one-to-one* or *injective* function), if for every $x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Example. Let $A \subseteq \mathbf{R}$ and $f : A \to \mathbf{R}$ be a strictly increasing function. Then f is injective.

Example. Which of the following is an injective function?

a) $f(x) = 3x + 1$ b) $g(x) = x^2$ c) $h(x) = \dfrac{3x+4}{x-2}$

Note: negation of injectivity is: there exist $x_1, x_2 \in A$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

Since proofs using $\neq$ are inconvenient (cannot apply rules similar to equations), we usually use the contrapositive statement to the definition of injectivity, that is: for every $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Use this formulation to check injectivity of functions in the example.

Note: Domain matters. If we consider $g : [0, \infty) \to \mathbf{R}$, it is injective.

Definition. Let $f : A \to B$ be a function. We call f an *surjection* (or *onto* or *surjective* function), if range(f) =codom(f). This is equivalent to: for every $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

Note: negation of surjectivity is: there exists a $y \in B$ such that for all $x \in A$, $f(x) \neq y$.

Example. Which of the following is a surjective function?

a) $f(x) = 3x + 1$ b) $g(x) = x^2$ c) $h(x) = \dfrac{3x+4}{x-2}$

Note: We can fix absence of surjectivity by altering the codomain.

$g : [0, \infty) \to [0, \infty)$ is surjective $\qquad$ $h : \{x \mid x \neq 2\} \to \{y \mid y \neq 3\}$ is surjective

Example. Let $f : \mathbf{Z} \to \mathbf{Z}$, $f(x) = 2x$. Show f is injective, but not surjective.

Example. Construct an $f : \mathbf{Z} \to \mathbf{Z}$ that is surjective, but not injective.

Definition. A function $f : A \to B$ is called a *bijection* (or *one-to-one and onto* or *bijective* function), if f is injective and surjective.

Example.

$f : \mathbf{R} \to \mathbf{R}$, $f(x) = 3x + 1$ is bijective

$g : [0, \infty) \to [0, \infty)$, $g(x) = x^2$ is bijective

$h : \{x \mid x \neq 2\} \to \{x \mid x \neq 3\}$, $h(x) = \dfrac{3x + 4}{x - 2}$ is bijective

Usually, if a function is not injective or surjective, this can be fixed by altering the domain or codomain.

Example. $\sin : \mathbf{R} \to \mathbf{R}$ is neither injective or surjective

Proposition. Let A be any set. Then there does not exist a surjection $A \to \mathcal{P}(\mathcal{A})$.
(A surjection $\mathcal{P}(\mathcal{A}) \to \mathcal{A}$ is easy to construct.)

Proof.

Note: A consequence of this is that there does not exist a “biggest set,” and that some infinite sets are “bigger” than others.