5.1 Operations on Sets

We work in a universal set U.

Recall: $x \in A$: x is an element of A
$x \notin A$: x is not an element of A
$\emptyset$ denotes a set with no elements, the *empty set* or *null set*

Definition. Let A and B be sets with elements from a universal set U. We define:

1) A is a subset of B, $A \subseteq B$, if for all $x \in U$, if $x \in A$, then $x \in B$.

2) A is equal to B, $A = B$, if for all $x \in U$, $x \in A$ if and only if $x \in B$.

We write $A \not\subseteq B$, $A \neq B$ for negations of those statements. We also write $A \subset B$, and say A is a proper subset of B, if $A \subseteq B$ and $A \neq B$.

Note. $A = B$ is equivalent to $A \subseteq B$ and $B \subseteq A$.

Example. Let
$U = \mathbf{N}$, $\quad A = \{1, 3, 5, 7\}$, $\quad B = \{n \in \mathbf{N} \mid n \text{ is odd and } n < 9\}$, $\quad C = \{n \in \mathbf{N} \mid n \text{ is odd}\}$.
State the relationships among these sets.

Example. Let $U = \mathbf{Z}, A = \{n \in \mathbf{Z} \mid\ 2 \mid n\}, B = \{n \in \mathbf{Z} \mid\ 4 \mid n\}$.
Show that $B \subset A$ and $A \not\subseteq B$.

Note. The negation of $(\forall x \in U)(x \in A \Longrightarrow x \in B)$ is $(\exists x \in U)(x \in A \text{ and } x \notin B)$.

We often use Venn diagrams to picture relationships between sets.

$A \subseteq B$

Definition. We define the following operations on sets. Let $A, B \subseteq U$.

Intersection of A and B
$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$

Union of A and B
$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$

Set difference of A and B
$A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}$

Complement of A
$A^c = \{x \in U \mid x \notin A\}$

Example. Let $U = \mathbf{R}$, $A = [5, \infty)$, $B = (-\infty, 7)$, $C = [4, 6]$. Find the following sets.

$A \cap B =$ $A \cap C =$ $A \cup B =$ $C - A =$

$B - C =$ $C - B =$ $A - B =$ $A^c =$

$B - A =$ $B \cup C =$ $B \cap C =$ $B \cap A^c =$

Definition. The *cardinality* $|A|$ of a finite set A is the number of elements in A.

Example. $|\{1, 3, 7, 8\}| =$ $\qquad\qquad$ $|\{\sqrt{8}, 4, 13, 2\sqrt{2}\}| =$

Definition. The *power set of* A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

$$\mathcal{P}(A) = \{X \subseteq U \mid X \subseteq A\}$$

Note: We always have $\emptyset \subseteq A$ and $A \subseteq A$. Therefore, $\emptyset \in \mathcal{P}(A)$ and $A \in \mathcal{P}(A)$.

Example. Let $A = \{a, b, c\}$.

$|A| =$ $\qquad\qquad$ $\mathcal{P}(A) =$

Note: In general, it is true that $|\mathcal{P}(A)| = 2^{|A|}$.

Note: For an element $a \in A$, $\{a\}$ and a are not the same thing.

5.2 Proving Set Relationships

We demonstrate how to prove claims about sets.

Proposition. Let $A, B \subseteq U$. Then $A - B = A \cap B^c$.

Proof.

Note that we can often just retrace our steps for the opposite inclusion.

Proposition. Let $A, B, C \subseteq U$. If $C \subseteq A - B$, then $B \cap C = \emptyset$.

Proof.

Definition. Sets A and B are *disjoint* if $A \cap B = \emptyset$.

Proposition. Let $A, B, C \subseteq U$. If $A \cup C = B \cup C$ and $A \cap C = B \cap C$, then $A = B$.

Proof.

5.3 Properties of Set Operations

Theorem. Let A, B and C be subsets of some universal set U. Then the following equalities or statements hold.

Empty set	$A \cap \emptyset = \emptyset$	$A \cup \emptyset = A$
Universal set	$A \cap U = A$	$A \cup U = U$
Idempotent Laws	$A \cap A = A$	$A \cup A = A$
Commutative Laws	$A \cap B = B \cap A$	$A \cup B = B \cup A$
Associative Laws	$(A \cap B) \cap C = A \cap (B \cap C)$ $(A \cup B) \cup C = A \cup (B \cup C)$	
Distributive Laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	
Basic Complement Laws	$(A^c)^c = A$	$A - B = A \cap B^c$
Empty Set Complements	$\emptyset^c = U$	$A - \emptyset = A$
Universal Set Complements	$U^c = \emptyset$	$A - U = \emptyset$
De Morgan's Laws	$(A \cap B)^c = A^c \cup B^c$ $(A \cup B)^c = A^c \cap B^c$	
General Subset Laws	$A \cap B \subseteq A$	$A \subseteq A \cup B$
	If $A \subseteq B$, then $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$	
Subsets and Complements	$A \subseteq B$ if and only if $B^c \subseteq A^c$	

Example. Prove the distributive law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Example. Prove: if $A \subseteq B$, then $A \cap C \subseteq B \cap C$.

Example. Prove De Morgan's law $(A \cup B)^c = A^c \cap B^c$.

Note. Proofs of set equalities heavily depend on logical equivalences we have proved before. For example, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ uses the logical equivalence $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$. De Morgan's laws for sets use De Morgan's logical equivalences.

Example. Use algebra of sets to show that $A - (A - B) = A \cap B$.

Example. Use algebra of sets to show that $(A - B) \cap (B - A) = \emptyset$.