4.1, 4.2 Mathematical Induction

Mathematical induction is a tool for proving statements about natural numbers or integers. We are usually proving

$$(\forall n \in \mathbf{N})\ P(n),\ \text{ where } P(n) \text{ is some open sentence in } n$$

The essential idea of mathematical induction is as follows:

Basis step: prove $P(1)$.
Inductive step: prove: for every $k \in \mathbf{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbf{N}$.

Why it works: if you have done both steps, then:
$P(1)$ is true by the basis step.
Since $P(1)$ is true, then $P(2)$ is true by the inductive step.
Since $P(2)$ is true, then $P(3)$ is true by the inductive step, etc.

It works like dominoes set up in a line: when the first one is knocked over, it knocks over the second one, which knocks over the third one, which knocks over the fourth one, etc.

Example. Show that for every $n \in \mathbf{N}$, $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.
In sigma notation: $\displaystyle\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$.

Theoretically, mathematical induction is framed like this:

Definition. A set $T \subseteq \mathbf{Z}$ is *inductive* if for every $k \in \mathbf{Z}$, if $k \in T$, then $k+1 \in T$.

Example. Which of the following sets are inductive?

$\{-3, -2, -1, 0, 1, 2, 3, \ldots\}$

$\{1, 3, 5, 7, \ldots\}$

$\{\ldots, -2, -1, 0, 1\}$

$\{6, 7, 8, 9, 10, \ldots\}$

Principle of Mathematical Induction. (An axiom of $\mathbf{N}$.) If $T \subseteq \mathbf{N}$ has the properties:
1) $1 \in T$ 2) for every $k \in \mathbf{N}$, if $k \in T$, then $k+1 \in T$ (that is, T is inductive)
Then $T = \mathbf{N}$.

We essentially prove that the truth set of $P(n)$ satisfies properties 1 and 2, therefore the truth set is $\mathbf{N}$.

Example. Show that $n! \geq 4^n$ for every $n \geq 9$ $(n \in N)$.

Note: this is a version of induction that does not start at 1.

Example. Show that $n^5 \leq 3^n$ for every $n \geq 11$ ($n \in N$).

Theorem. Each natural number greater than 1 is either a prime or a product of prime numbers.

Let $P(n) =$ "n is a prime or a product of primes".

Note: this is a version of induction where we assume $P(m), P(m+1), \ldots, P(k)$ (rather than only $P(k)$), and show $P(k+1)$ follows from the assumption.