Mathematical Reasoning — Lecture notes
MAT 312, Spring 2023 — D. Ivanšić

2.1, 2.2 Logical Operators, Equivalent Statements

If we have two or more statements, we can build more complex statements using the statements and a **logical operator** (or **connective**). The common logical operators are *not, and, or, if–then.*

Operator	In words	In symbols
Negation	not P	$\neg P$
Conjunction	P and Q	$P \wedge Q$
Disjunction	P or Q	$P \vee Q$
Implication	If P, then Q	$P \Longrightarrow Q$

Let P and Q be the statements:

P: The house has a 3-car garage.
Q: The house has 4 bathrooms.

With these statements in mind as examples, fill out the truth tables below:

P	not P
T	
F	

P	Q	P and Q
T	T	
T	F	
F	T	
F	F	

P	Q	P or Q
T	T	
T	F	
F	T	
F	F	

Definition. Two expressions X and Y are called **logically equivalent** if they have the same truth value for all possible combinations of truth values of variables appearing in the two expressions. In this case, we write $X \equiv Y$ and say "X is equivalent to Y."

Example. Consider the "house" statements from above to convince yourself of de Morgan's laws:

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$
$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

Now use truth tables to establish the same:

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T								
T	F								
F	T								
F	F								

Example. Use truth tables to establish one distributive law: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$. The other is $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

Example. Let P and Q be the statements:

P: It is raining.
Q: There are clouds in the sky.

Form the statements below. Which of them mean the same thing as $P \implies Q$?

$P \implies Q$

$Q \implies P$

$\neg Q \implies \neg P$

$\neg P \implies \neg Q$

$\neg P \vee Q$

$P \wedge \neg Q$

$\neg(P \wedge \neg Q)$

Recall the truth table for $P \implies Q$ to fill out the following truth tables. Compare the statements $P \implies Q$, $Q \implies P, \ldots, \neg(P \wedge \neg Q)$. Which have the same truth values when given the same input for P and Q? Does it agree with your answers above?

P	Q	$\neg P$	$\neg Q$	$P \implies Q$	$Q \implies P$	$\neg Q \implies \neg P$	$\neg P \implies \neg Q$	$\neg P \vee Q$	$P \wedge \neg Q$	$\neg(P \wedge \neg Q)$
T	T									
T	F									
F	T									
F	F									

More on the conditional statement: if P, then Q.

Other ways to say it:

As an equivalent statement: $P \Longrightarrow Q \;\equiv\; \neg Q \Longrightarrow \neg P \;\equiv\; \neg P \vee Q \;\equiv\; \neg(P \wedge \neg Q)$

Q if P

Graphically:

P implies Q

Whenever P is true, Q is true

Q is true whenever P is true

Q is necessary for P

P is sufficient for Q

P only if Q

The biconditional statement $P \Longleftrightarrow Q$

The statement $P \Longleftrightarrow Q$ is read as "P if and only if Q," so

$$\text{it consists of two statements connected by "and"} \begin{cases} P \text{ if } Q\ (Q \Longrightarrow P) \\ P \text{ only if } Q\ (P \Longrightarrow Q) \end{cases}$$

Therefore, we can write $P \Longleftrightarrow Q \equiv (P \Longrightarrow Q) \wedge (Q \Longrightarrow P)$

Truth table:

P	Q	$P \Longrightarrow Q$	$Q \Longrightarrow P$	$(P \Longrightarrow Q) \wedge (Q \Longrightarrow P)$
T	T			
T	F			
F	T			
F	F			

Note: $P \Longleftrightarrow Q$ means P and Q are either both true or both false.

Example: The light is on $\Longleftrightarrow$ the switch on the light is on
(assuming normal circumstances)

Example: A number is divisible by 4 $\Longleftrightarrow$ the last two digits of the number is a number divisible by 4

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2.3 Open Sentences and Sets

Definition. A *set* is a collection of objects.

Examples.

$A =$ set of integers divisible by $3 = \{\dots, -6, -3, 0, 3, 6, \dots\}$

$B =$ set of real solutions to equation $x^2 = 7 \quad = \{x \in \mathbf{R} \mid x^2 = 7\} = \{ \qquad \qquad \}$

$C =$ set of all real numbers x for which $x^2 - 3x - 10 \leq 0 = \{x \in \mathbf{R} \mid x^2 - 3x - 10 \leq 0\} =$

$D =$ set of all integers x for which $x^2 - 3x - 10 \leq 0 = \{x \in \mathbf{Z} \mid x^2 - 3x - 10 \leq 0\} =$

Notice the ways of writing sets:

enumerating all elements, or enumerating enough elements so pattern is clear (roster method)

set builder notation: $\{x \in U \mid P(x)\}$, where $P(x)$ is some sentence in x,
U is a universal set
(set that makes sense in the context)

Notation: $y \in A$: y is an element of A
$y \notin A$: y is not an element of A
$\emptyset$ denotes a set with no elements, the *empty set* or *null set*

Example. Write whether the following are or are not elements of sets A, B, C, D above:

$4 \quad A \qquad -18 \quad A \qquad 0 \quad B \qquad \sqrt{7} \quad B$

$1.17 \quad C \qquad 6 \quad C \qquad 1.17 \quad D \qquad 1 \quad D \qquad -3 \quad D$

Definition. Two sets A and B are equal if they have precisely the same elements.

A set A is a *subset* of a set B ($A \subseteq B$) if every element of A is an element of B. We say A is *contained* in B. We can also say B contains A: $B \supseteq A$.

Example. Write the correct relationship between sets ($=$, $\neq$, $\subseteq$, $\supseteq$)

$[-2, 5] \qquad \{-2, -1, 0, 1, 2, 3, 4, 5\} \qquad\qquad \{1, 3, 5, 5\} \qquad \{1, 3, 5\} \qquad\qquad \emptyset \qquad \mathbf{N}$

$\{1, 3, 5, 7, 9, 11, \dots\} \qquad \{x \in \mathbf{N} \mid x = 2k + 1 \text{ for some } k \in \mathbf{N}\}$

$\{x \in \mathbf{N} \mid x = 4k \text{ for some } k \in \mathbf{N}\} \qquad \{x \in \mathbf{N} \mid \text{the last two digits of } x \text{ form a number divisible by } 4\}$

Example. Sentences such as

$x^2 - 7 = 0$ or $x^2 - 3x - 10 \leq 0$ or $\sqrt[4]{x}$ is a real number

are not statements — since we don't know what x is, we cannot determine their truth value.

Such statements are called *open sentences*. The above have form $P(x)$, that is, they have one variable, but they may have more.

Example. $3x - y = z$ has form $P(x, y, z)$ — it has three variables.

Putting in specific values for the variables makes the open sentences true or false, so they become statements.

Definition. An *open sentence* (or *predicate* or *propositional function*) is a sentence $P(x_1, x_2, \ldots, x_n)$ involving variables $x_1, x_2, \ldots, x_n$ with the property that when specific values from the universal set are assigned to $x_1, x_2, \ldots, x_n$, the resulting sentence is either true or false, so is a statement.

Example. Let $U = \{\text{Alice, Bernie, Charlene}\}$. Consider these predicates.

$3x + 1 = 7x - 5$

x works at a pizzeria

Example. Let $U = \mathbf{R}$. Which x makes the sentence true?

$3x + 1 = 7x - 5$

Definition. The truth set of an open sentence with one variable is the collection of all objects in the universal set that, when substituted for the variable, make the open sentence a true statement.

Example. Consider the open sentences and universal sets below. What is the truth set of each open sentence?

$3x + 1 = 7x - 5$ $U = \mathbf{R}$ $U = \mathbf{Q}$ $U = \mathbf{Z}$

$\sqrt[4]{x}$ is a number in the universal set $U = \mathbf{R}$ $U = \mathbf{Q}$ $U = \mathbf{Z}$

$x^2 - 3x - 10 \leq 0$ $\qquad U = \mathbf{R}$ $\qquad U = \mathbf{Q}$ $\qquad U = \mathbf{Z}$

Note. The truth set of $P(x)$ is $\{x \in U \mid P(x)\}$.

2.4 Quantifiers and Negations

Example. Consider the sentences. Are they statements?

For every $x \in \mathbf{R}$, $x^2 + 5x + 10 \geq 0$.

There exists an $x \in \mathbf{R}$, such that $x^2 + 5x + 10 \geq 0$

For every $x \in \mathbf{R}$, $x^3 + 3x^2 + 1 \geq 0$.

There exists an $x \in \mathbf{R}$, such that $x^3 + 3x^2 + 1 \geq 0$.

The statements are formed by using open sentences in a variable and phrases

"for every x in a universal set" (universal quantifier)
"there exists an x in a universal set (existential quantifier)

The quantifiers turn an open sentence into a statement.

Notation: $(\forall x)\, P(x)$: for every x, $P(x)$
$(\exists x)\, P(x)$: there exists an x such that $P(x)$.

Note: $(\forall x)\, P(x) \equiv$ the truth set of $P(x)$ is the universal set
$(\exists x)\, P(x) \equiv$ the truth set of $P(x)$ is not empty

Example. Negate the statements.

Every Batman has a mask.

There is a rotten apple in the barrel.

For every $x \in \mathbf{R}$, $3^x > 2^x$.

There exists an $x \in \mathbf{R}$ such that $x^2 + 4 = 2x$.

The previous examples should give you an intuitive understanding of this negation principle.

Theorem. For an open sentence $P(x)$

$$\neg(\forall x)\ P(x) \equiv (\exists x)\ \neg P(x)$$
$$\neg(\exists x)\ P(x) \equiv (\forall x)\ \neg P(x)$$

Proof.

Example. Definitions involve quantifiers, so we now know how to negate them. Negate the following definitions.

An integer x is even if there exists an integer k such that $x = 2k$.

A real number x is rational if there exist integers a, b such that $x = \dfrac{a}{b}$.

Example. Working with multiple quantifiers.

$$\begin{array}{l}(\exists y \in \mathbf{R})(x^2 + y^2 = 25) \\ (\forall y \in \mathbf{R})(x^2 + y^2 = 25)\end{array} \text{ are not statements (what is } x\text{?) but open sentences in } x \left\{ \begin{array}{l} P(x) \\ Q(x) \end{array} \right.$$

Check whether they are true or false for $x = 3$ and $x = 7$.

Determine the truth sets of the open sentences $P(x)$ and $Q(x)$.

$(\exists y \in \mathbf{R})(x^2 + y^2 = 25)$

$(\forall y \in \mathbf{R})(x^2 + y^2 = 25)$

We can form statements using quantifiers and $P(x)$ and $Q(x)$.

$(\forall x \in \mathbf{R})\ P(x) \equiv (\forall x \in \mathbf{R})(\exists y \in \mathbf{R})(x^2 + y^2 = 25)$
$(\exists x \in \mathbf{R})\ P(x) \equiv (\exists x \in \mathbf{R})(\exists y \in \mathbf{R})(x^2 + y^2 = 25)$
$(\forall x \in \mathbf{R})\ Q(x) \equiv (\forall x \in \mathbf{R})(\forall y \in \mathbf{R})(x^2 + y^2 = 25)$
$(\exists x \in \mathbf{R})\ Q(x) \equiv (\exists x \in \mathbf{R})(\forall y \in \mathbf{R})(x^2 + y^2 = 25)$

For the first and third statements, write the statement in words and negate it using quantifiers. Write the negated statements in words.