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| **Advanced Calculus 1 — Exam 1**<br>**MAT 525/625, Fall 2023 — D. Ivanšić** |

**Name:** ______________________________

*Show all your work!*

*Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.*

**Theory 1.** (3pts) Define the infimum of a nonempty subset $S$ of $\mathbf{R}$.

**Theory 2.** (3pts) State the order properties of $\mathbf{R}$.

**Theory 3.** (3pts) State the lemma that characterizes when a number $u$ is the supremum of a set $S$, but does not use $\varepsilon$.

## Type A problems (5pts each)

**A1.** Show using Mathematical Induction that $1 + x + \cdots + x^n = \dfrac{x^{n+1} - 1}{x - 1}$ for any $x \in \mathbf{R}$, $x \neq 1$.

**A2.** Let $A$ and $B$ be sets, where $A$ is denumerable and $B$ is finite. Show that $A \setminus B$ is denumerable.

**A3.** If $a, b \in \mathbf{R}$, use only the axiomatic algebraic properties of $\mathbf{R}$ to show that $\frac{1}{\frac{1}{a}} = a$ and $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$.

**A4.** Let $S$ be nonempty and bounded. Show that 1) $\inf S \leq \sup S$, and 2) if $\inf S = \sup S$, then $S$ has only one element.

**A5.** Show using the triangle inequality: for all $x, y \in \mathbf{R}$, $||x| - |y|| \leq |x - y|$.

**A6.** For these subsets of $\mathbf{R}$, if they exist, find a lower bound of $S$, an upper bound of $S$, $\inf S$ and $\sup S$. There is no need to justify.
a) $S = \{x \in \mathbf{Q} \mid \sqrt{2} < x < \sqrt{3}\}$ b) $S = \left\{\frac{1}{n} + \frac{1}{n^2} \mid n \in \mathbf{N}\right\}$

## Type B problems (8pts each)

**B1.** Show that the set $\{\frac{1}{n} - \frac{1}{m} \mid m, n \in \mathbf{N}\}$ is denumerable.

**B2.** If $a, b \in \mathbf{R}$, use only the axiomatic algebraic properties of $\mathbf{R}$ to show that $(-a)(-b) = ab$, *without* proving $(-1)a = -a$ first.

**B3.** Show by induction on $n$: every number of form $2^{2n-1}m$, where $m, n \in \mathbf{N}$ and $m$ is odd has an irrational square root. (Note the predicate $P(n)$ is: for every odd $m \in \mathbf{N}$, the number $2^{2n-1}m$ has an irrational square root. Basis is more involved, step is easy.)

**B4.** Determine and sketch the set of points in the plane satisfying $2|x| - |y| \geq 4$.

**B5.** Consider the subset $S$ of $\mathbf{R}$, $S = \left\{ n \cdot \frac{1+(-1)^n}{2} + \frac{1}{n} \mid n \in \mathbf{N} \right\}$. If they exist, find a lower bound of $S$, an upper bound of $S$, $\inf S$ and $\sup S$. Prove the details, including nonexistence of any of the quantities.

**B6.** Let $S \subset [0, \infty)$ be a nonempty set. Show that $\inf S^2 = (\inf S)^2$. (As expected, $S^2 = \{s^2 \mid s \in S\}$.)

**B7.** Let $a, b \in \mathbf{R}$, $a < b$. Show that the interval $(a, b)$ contains a rational number of the form $\frac{m}{2^n}$, for some $m \in \mathbf{Z}$ and $n \in \mathbf{N}$.

## Type C problems (12pts each)

**C1.** Let $\alpha \in \mathbf{R}$ be such that $\alpha^3$ is rational, but $\alpha$ and $\alpha^2$ are not. Show: if $s, t, u \in \mathbf{Q}$ are such that $s + t\alpha + u\alpha^2 = 0$, then $s = t = u = 0$. *Hint: multiply by a number of form $x + y\alpha$, $x, y \in \mathbf{Q}$, and choose $x$ and $y$ wisely.*

**C2.** Let $\mathcal{C} = \{A \subseteq \mathbf{Q} \mid A \subseteq (0, 1)\}$. Show that this collection of subsets of $\mathbf{Q}$ is uncountable. *Hint: start by showing the map $\mathcal{C} \to \mathbf{R}$, $A \mapsto \sup A$ is surjective.*

**Advanced Calculus 1 — Exam 2**
**MAT 525/625, Fall 2023 — D. Ivanšić**

**Name:** ______________________________

*Show all your work!*

*Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.*

**Theory 1.** (3pts) Define when a sequence tends to $-\infty$.

**Theory 2.** (3pts) Define a Cauchy sequence.

**Theory 3.** (3pts) State the Bolzano-Weierstrass Theorem.

## Type A problems (5pts each)

**A1.** Use the definition of the limit to show $\lim \frac{n-1}{n^2+2n} = 0$.

**A2.** Find $\lim \left(1 + \frac{1}{2n}\right)^{n-1}$.

**A3.** Use the "ratio test" or other method to find the limit of the sequence $\frac{n^{3n}}{(3n)!}$.

**A4.** Determine if the sequence $\left(\sin \frac{(2n+1)\pi}{4}\right)$ converges. Justify your answer with known theorems.

**A5.** Show: if $\lim x_n = x$ and $x < 0$, then there is a $K \in \mathbf{N}$ such that $x_n < 0$ for all $n \geq K$.

**A6.** Prove the extended limit law $L \cdot \infty = \infty$ for $L > 0$. That is, use the definition to show: if $\lim x_n = L \in \mathbf{R}$, $L > 0$ and $\lim y_n = \infty$, then $\lim(x_n y_n) = \infty$.

## Type B problems (8pts each)

**B1.** Find $\lim \sqrt[n]{(1 + \frac{1}{n})^n + 2^n}$.

**B2.** Let the sequence $x_n$ be recursively given by: $x_1 = 6$, $x_{n+1} = \sqrt{5 + 2x_n}$. Show that this sequence converges and find its limit.

**B3.** Let $x_n = 1 + b + b^2 + \cdots + b^n$, $|b| < 1$. Show that $x_n$ is a contractive sequence and find $\lim x_n$. Note that the obvious recursive relation $x_{n+1} = x_n + b^{n+1}$ does not help. For example, use $bx_n = \ldots.$

**B4.** Let $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$. Show directly that $(x_n)$ is a Cauchy sequence. The inequality $\frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$ will be helpful.

**B5.** Prove the other version of the "ratio test:" for a sequence of positive numbers $(x_n)$, if $\lim \frac{x_{n+1}}{x_n} = L$ and $L \in \mathbf{R}$ and $L > 1$, then $\lim x_n = \infty$.

**B6.** Let $(x_n)$ be a bounded sequence and $w = \inf\{x_n \mid n \in \mathbf{N}\}$. Suppose that for every $v > w$, $x_n < v$ for infinitely many indices $n$. (Put another way: for every $v > w$, the set $\{n \in \mathbf{N} \mid x_n < v\}$ is infinite.) Show there exists a subsequence $(x_{n_k})$ such that $\lim x_{n_k} = w$.

## TYPE C PROBLEMS (12PTS EACH)

**C1.** Let the sequence $x_n$ be recursively given by: $x_1 > 0$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{7}{x_n}\right)$.
a) Show for any $a > 0$: $\frac{1}{2}\left(a + \frac{7}{a}\right) \geq \sqrt{7}$.
b) Show that $x_n$ is a decreasing sequence for $n \geq 2$ (part a) will help) and show it is bounded below.
c) Find $\lim x_n$.

**C2.** Let $x_n = \sqrt[n]{n!}$.
a) Show $(x_n)$ is increasing.
b) Show $n! \geq k!k^{n-k}$ for every $k = 1, 2, \ldots, n$.
c) Fixing a $k$, use b) to get an inequality that will give you the limit of $(x_n)$.

| Advanced Calculus 1 — Exam 3<br>MAT 525/625, Fall 2023 — D. Ivanšić | Name: ____________________<br>*Show all your work!* |
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*Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.*

**Theory 1.** (3pts) Let $f : A \to \mathbf{R}$ and let $c$ be a cluster point of $A$. Define what $\lim_{x\to c} f(x) = L$ means.

**Theory 2.** (3pts) State the limit law for products.

**Theory 3.** (3pts) State the Squeeze Theorem for limits.

## Type A problems (8pts each)

**A1.** Find the limits, if they exist (just a computation is expected): a) $\lim_{x\to c} \dfrac{\frac{1}{x} - \frac{1}{c}}{x - c}$ b) $\lim_{x\to 1} \dfrac{x^2 + x - 2}{x - \sqrt{x}}$.

**A2.** Does $\lim_{x\to 0} x^2 \left(3 + \sin \dfrac{1}{x}\right)$ exist? If yes, find it, if not, justify.

**A3.** Show that $\lim_{x\to 0} \dfrac{1}{x} \cos \dfrac{1}{x}$ does not exist.

**A4.** Let $f : \mathbf{R} \to \mathbf{R}$ and let $\lim_{x\to c} f(x) = L > 0$. Show that there is a $\delta$-neighborhood of $c$ so that $f(x) > 0$ for all $x \in V_\delta(c)$, $x \neq c$. Is $f(c) > 0$ also true?

**A5.** Use the definition to show that if $\lim_{x\to 0} f(x) = L$, then $\lim_{x\to c} f(x - c) = L$. (This justifies the following "substitution" when finding limits: $\lim_{x\to c} f(x) = [u = x - c] = \lim_{u\to 0} f(u + c)$.)

## Type B problems (8pts each)

**B1.** Use the definition to show that $\lim_{x\to c} x^4 = c^4$.

**B2.** Use the definition to show that $\lim_{x\to \frac{1}{4}} \dfrac{1}{x} = 4$.

**B3.** Suppose that $\lim_{x\to c}(f(x) - g(x)) = L$ and $\lim_{x\to c}(f(x)^2 + g(x)^2) = M$ for some $L, M \in \mathbf{R}$. Show that $\lim_{x\to c} f(x)g(x)$ exists and express it using $L$ and $M$.

**B4.** Find all the cluster points (in $\mathbf{R}$) of the set $\{\frac{1}{n} \mid n \in \mathbf{N}\}$. (There aren't many!) You do not need to write a detailed proof, but justify your answer with a picture and some good words. Justify also why certain points are *not* cluster points of the set.

**B5.** Suppose $f, g : \mathbf{R} \to \mathbf{R}$, and $\lim_{x\to c} g(x) = M$, $\lim_{x\to M} f(x) = L$ and there is a $\delta$ so that $g(x) \neq M$ for all $x \in V_\delta(c)$. Show that $\lim_{x\to c} f(g(x)) = L$.

**B6.** Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = \begin{cases} 2x+3, & \text{if } x \in \mathbf{Q} \\ 5-x, & \text{if } x \notin \mathbf{Q} \end{cases}$
Find all the numbers $c$ for which $\lim_{x \to c} f(x)$ exists, and prove the limit exists. Justify why $\lim_{x\to c} f(x)$ does not exist for the other numbers.

## Type C problems (12pts each)

**C1.** Recall that we proved that the sequence $e_n = \left(1 + \frac{1}{n}\right)^n$ was increasing and bounded. Using the same computation with slight modifications, we can prove that for every $x \geq 1$, the sequence $\left(1 + \frac{x}{n}\right)^n$ is increasing and bounded and hence converges to some number $f(x)$. Assuming this fact, do the following problems.

a) For $0 < x < 1$, show that $\left(1 + \frac{x}{n}\right)^n$ converges as follows: Let $M \in \mathbf{N}$ be such that $Mx \geq 1$, then $\left(1 + \frac{Mx}{n}\right)^n$ converges by the above. Then $\left(1 + \frac{x}{n}\right)^n = \left(1 + \frac{Mx}{Mn}\right)^{\cdots}$. Continue with the algebra to show this sequence converges, expanding the definition of $f(x)$ to $[0,1)$.

b) Now let $x < 0$, $x = -w$. Show that $\left(1 + \frac{x}{n}\right)^n$ converges by rewriting:

$$\left(1 + \frac{x}{n}\right)^n = \left(1 - \frac{w}{n}\right)^n = \frac{1}{\left(1 + \frac{w}{n-w}\right)^n}$$

Now let $M \in N$ be such that $M - 1 \leq w < M$. Show that

$$\frac{1}{1 + \frac{w}{n-(M-1)}} \geq \frac{1}{1 + \frac{w}{n-w}} > \frac{1}{1 + \frac{w}{n-M}}$$

and use the squeeze theorem and the results from above to show that $\dfrac{1}{\left(1 + \frac{w}{n-w}\right)^n}$ converges to $\dfrac{1}{f(w)}$. Therefore, we can expand the definition of $f$ to $(-\infty, 0)$, so we have that $f(x) = \lim \left(1 + \frac{x}{n}\right)^n$ is always defined and that $f(-x) = \dfrac{1}{f(x)}$.

**C2.** For the function $f(x) = \lim \left(1 + \frac{x}{n}\right)^n$, whose existence was proven above, show that $f(q) = e^q$ for every $q \in \mathbf{Q}$ as follows:
a) Show $f(mx) = f(x)^m$ for every $m \in \mathbf{N}$, and note it's obvious for $m = 0$. Use it to show that $f(m) = e^m$ for every $m \in \mathbf{N}$ and $m = 0$.
b) Use a) to show $f(q) = e^q$ for every $q \in \mathbf{Q}$, $q > 0$.
c) Use the last sentence from the problem above to show that $f(q) = e^q$ for every $q \in \mathbf{Q}$, $q < 0$.