Advanced Calculus 1 — Exam 3 **MAT 525/625, Fall 2019 — D. Ivanšić**	**Name:** ____________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define when a function $f : A \to \mathbf{R}$ is bounded on a neighborhood of c, a cluster point of A.

Theory 2. (3pts) Define what $\lim_{x\to\infty} f(x) = L$ means.

Theory 3. (3pts) State the limit law for quotients.

Type A problems (8pts each)

A1. Find the limits, if they exist (just a computation is expected): a) $\displaystyle\lim_{x\to-2} \frac{x^2-3x-10}{x+2}$ b) $\displaystyle\lim_{x\to\infty} (\sqrt{x+7}-\sqrt{x+3})$.

A2. Does $\displaystyle\lim_{x\to\infty} \frac{\sin x + \cos x}{x}$ exist? If yes, find it, if not, justify.

A3. Show that $\displaystyle\lim_{x\to-1} \cos\frac{1}{x+1}$ does not exist.

A4. Use the definition to show that $\lim_{x\to3} (2x+1) = 7$.

A5. Let $f(x) \geq 0$ and $L = \lim_{x\to c} f(x)$. Use the definition to show that $L \geq 0$, by showing that $L \geq -\epsilon$ for every $\epsilon > 0$.

Type B problems (8pts each)

B1. Use the definition to show that $\displaystyle\lim_{x\to c} \frac{1}{x^2+4} = \frac{1}{c^2+4}$.

B2. Use the definition to prove the extended limit law $\infty+\infty=\infty$, that is, if $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$, show that $\lim_{x\to c}(f(x)+g(x)) = \infty$.

B3. Suppose $f(x), g(x) > 0$, $\lim_{x\to c} f(x)g(x) = L$ and $\lim_{x\to c} f(x) = 0$. Show that $\lim_{x\to c} g(x) = \infty$.

B4. Find all the cluster points (in $\mathbf{R}$) of the set $(-\sqrt{2}, \sqrt{2}) \cap \mathbf{Q}$. You do not need to write a detailed proof, but justify your answer with a picture and a few words. Justify also why certain points are *not* cluster points of the set.

B5. Use the definition and properties of the sine function to show that $\lim_{x\to c} \sin x = \sin c$. You will find the formula $\sin u - \sin v = 2\sin\frac{u-v}{2}\cos\frac{u+v}{2}$ useful.

B6. Suppose $f, g : (a,\infty) \to \mathbf{R}$, and $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to L} g(x) = \infty$. Show that $\lim_{x\to\infty} f(g(x)) = L$.

B7. Use a picture of the unit circle and areas to show that $\lim_{\theta \to 0} \cos\theta = 1$ by showing that $\lim_{\theta \to 0}(1 - \cos\theta) = 0$.

Type C problems (12pts each)

C1. Let $f : (0, \infty) \to \mathbf{R}$, $f(x) = 0$ if x is irrational, $f(x) = \frac{1}{n}$ if x is a rational number represented by $\frac{m}{n}$ in reduced form. Show that for every c, $\lim_{x \to c} f(x) = 0$.
Hint: show that in an open interval of width $\frac{1}{n}$ there are only finitely many rational numbers x so that $f(x) > \frac{1}{n}$ and take it from there.