Advanced Calculus 1 — Exam 2
MAT 525/625, Fall 2019 — D. Ivanšić

Name: ____________________________

Show all your work!

Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define when the limit of a sequence (x_n) is x.

Theory 2. (3pts) State the squeeze theorem for sequences.

Theory 3. (3pts) State the Cauchy Convergence Theorem.

Type A problems (5pts each)

A1. Use the definition of the limit to show $\lim \frac{n+3}{5n+4} = \frac{1}{5}$.

A2. Find $\lim \left(1 + \frac{1}{n+1}\right)^{n-1}$.

A3. Find $\lim \sqrt[n]{n^2 + 4n + 7}$.

A4. Use the "ratio test" or other method to find the limit of the sequence $\frac{b^n}{n!}$, $b > 0$.

A5. Prove the extended limit law $L + \infty = \infty$. That is, use the definition to show: if $\lim x_n = L \in \mathbf{R}$ and $\lim y_n = \infty$, then $\lim(x_n + y_n) = \infty$.

A6. Use the definition to show that the sequence $x_n = \frac{n}{n+6}$ is Cauchy.

Type B problems (8pts each)

B1. Let the sequence x_n be recursively given by: $x_1 = 1$, $x_{n+1} = 3 + \sqrt{2x_n + 1}$. Show that this sequence converges and find its limit.

B2. Prove the limit law $\lim(x_n \cdot y_n) = \lim x_n \cdot \lim y_n$.

B3. Let $a > 0$ and $b > 1$. Find $\lim \frac{n^a}{b^n}$.

B4. Show that if a sequence of positive numbers (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim x_{n_k} = \infty$.

B5. Let the sequence x_n be recursively given by: $x_1 > 0$, $x_{n+1} = \frac{1}{3+x_n}$. Show that this sequence is contractive and find its limit.

B6. Suppose that for two positive sequences (x_n) and (y_n), $\lim x_n y_n = \infty$ and (x_n) is bounded. Show that $\lim y_n = \infty$.

B7. Let $\lim x_n = x$ and $\lim y_n = y$, where $x < y$, $x, y \in \mathbf{R}$, and let $z_n = \max\{x_n, y_n\}$. Show that $\lim z_n = y$.

Type C problems (12pts each)

C1. Let (x_n) be a bounded sequence that does not converge to x. (It does not mean it converges to some other number.) Show (x_n) has a subsequence that converges to a number y, where $y \neq x$.

C2. Let $x_n = \sqrt[n]{n!}$.
a) Show (x_n) is increasing.
b) Show $n! \geq k!k^{n-k}$ for every $k = 1, 2, \ldots, n$.
c) Fixing a k, use b) to get an inequality that will give you the limit of (x_n).