Advanced Calculus 1 — Exam 1
MAT 525/625, Fall 2019 — D. Ivanšić

Name: ______________________________

Show all your work!

Do all the theory problems. Then do five problems, at least one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define a denumerable set.

Theory 2. (3pts) State the multiplicative properties of $\mathbf{R}$. That is, state the four algebraic properties of $\mathbf{R}$ that deal only with multiplication.

Theory 3. (3pts) State the Completeness Property of $\mathbf{R}$.

Type A problems (5pts each)

A1. Show using Mathematical Induction that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. (Here i is the imaginary unit, $i^2 = -1$, and recall that $\sin(x+y) = \sin x\cos y + \cos x\sin y$ and $\cos(x+y) = \cos x\cos y - \sin x\sin y$).

A2. Let A and B be sets, where A is finite. Show: $A \cup B$ is finite if and only if B is finite. You may use the statement that the union of two disjoint finite sets is finite.

A3. If $a, b \in \mathbf{R}$, use only the axiomatic algebraic properties of $\mathbf{R}$ to show that $-(a+b) = (-a) + (-b)$, and $a \cdot (-b) = -ab$.

A4. Show that for all $x, y, z \in \mathbf{R}$, $||x-y| - |z-y|| \leq |x-z|$.

A5. If they exist, find a lower bound of S, an upper bound of S, $\inf S$ and $\sup S$. There is no need to justify. a) $S = \{x \mid x < 0\}$ b) $S = \left\{\frac{n-1}{n} \mid n \in \mathbf{N}\right\}$

Type B problems (8pts each)

B1. Show that the collection of two-element subsets of $\mathbf{N}$ is denumerable.

B2. Recall that we set $\frac{a}{b} = a \cdot \frac{1}{b}$. Use only the axiomatic algebraic properties of $\mathbf{R}$ to show the rule for adding fractions: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$. (You will need to prove an intermediate lemma.)

B3. Show directly that $\sqrt{8}$ is not a rational number, that is, without resorting to knowledge that $\sqrt{2}$ is not rational.

B4. Determine and sketch the set of points in the plane satisfying $|x| - 2|y| \leq 4$.

B5. Consider the subset S of $\mathbf{R}$, $S = \{x \in \mathbf{Q} \mid x > \pi\}$. If they exist, find a lower bound of S, an upper bound of S, $\inf S$ and $\sup S$. Prove the details, including nonexistence of any of the quantities.

B6. Let $S \subset (0, \infty)$ be a nonempty set that is bounded above. Show that $\inf \frac{1}{S} = \frac{1}{\sup S}$. (As expected, $\frac{1}{S} = \{\frac{1}{s} \mid s \in S\}$.)

Type C problems (12pts each)

C1. Show that the set of all functions $\mathbf{N} \to \{0,1\}$ is uncountable. *Hint: any function* $a : \mathbf{N} \to \{0,1\}$ *may be thought of as a sequence of 0's and 1's.*

C2. Let S be a bounded nonempty subset of $\mathbf{R}$.
a) Show by example that, in general, $\sup |S| \neq |\sup S|$.
b) Figure out the correct expression for $\sup |S|$ and prove it. Some homework statements may be helpful here, and you may use them without proof.