Advanced Calculus 1 — Problem Farm 1
MAT 525/625 Fall 2023 — D. Ivanšić

Sets and Functions

Type A problems (5pts each)

In **A1–A4** you do not need to write a formula or prove anything, just illustrate what the function does with an arrow diagram f rom which the pattern is clear.

A1. Give an example of a function $f : \mathbf{Z} \to \mathbf{Z}$ that is a bijection, but not the identity. Give an example of a function $f : \mathbf{N} \to \mathbf{N}$ that is a bijection, but not the identity.

A2. Give an example of a function $f : \mathbf{N} \to \mathbf{N}$ that is injective, but not surjective. Give an example of a function $f : \mathbf{Z} \to \mathbf{Z}$ that is injective, but not surjective.

A3. Give an example of a function $f : \mathbf{N} \to \mathbf{N}$ that is surjective, but not injective. Give an example of a function $f : \mathbf{Z} \to \mathbf{Z}$ that is surjective, but not injective.

A4. Give an example of a function $f : \mathbf{N} \to \mathbf{Z}$ that is surjective, but not injective.

A5. Give an example (formula) of a function $f : \mathbf{R} \to \mathbf{R}$ that is surjective, but not injective. Justify with graph. Give an example (formula) of a function $f : \mathbf{R} \to \mathbf{R}$ that is injective, but not surjective. Justify with graph.

A6. Give an example of a function $f : A \to B$, where both A and B are finite sets (i.e., you can draw a picture with blobs), and sets $E, F \subset A$, for which $f(E \cap F) \neq f(E) \cap f(F)$.

A7. Let functions $g : X \to Y$ and $f : Y \to Z$ be injective. Show that $f \circ g$ is injective.

A8. Let functions $g : X \to Y$ and $f : Y \to Z$ be surjective. Show that $f \circ g$ is surjective. Note that the consequence of **A7** and **A8** is that the composite of bijections is bijective.

A9. Let $f(x) = \sin x$.
a) Determine the direct image $f(A)$, if $A = \left(\frac{\pi}{4}, \frac{5\pi}{6}\right)$.
b) Determine the inverse image $f^{-1}(B)$, if $B = \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$.

Type B problems (8pts each)

B1. Let $p : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be given by $p(x, y) = x$.
a) Is p injective? Surjective?
b) Find $p^{-1}([-1, 5))$, and draw it in the plane.

B2. Let $p : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be given by $p(x, y) = 2x + y$.
a) Is p injective? Surjective?
b) Find $p^{-1}([-1, 5))$, and draw it in the plane.

B3. Let $A_n = \{n, n+1, \dots\}$ be subsets of $\mathbf{N}$. Determine $\displaystyle\bigcup_{n=1}^{\infty} A_n$ and $\displaystyle\bigcap_{n=1}^{\infty} A_n$ and justify your answers.

B4. Let $A_n = \{k \in \mathbf{N} \mid k \text{ divides } n\}$ be subsets of $\mathbf{N}$. For example, $A_{12} = \{1, 2, 3, 4, 6, 12\}$ and $A_{14} = \{1, 2, 7, 14\}$. Determine $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ and justify your answers.

B5. For $n \geq 2$, $n \in \mathbf{N}$, let $A_n = \left(-1 + \frac{1}{n}, 1 - \frac{1}{n}\right)$ be intervals in $\mathbf{R}$. Determine $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ and justify your answers.

Type C problems (12pts each)

C1. For $n \in \mathbf{N}$, let $A_n = [n, n^2]$ be intervals in $\mathbf{R}$. Determine $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ and justify your answers.

Advanced Calculus 1 — Problem Farm 2
MAT 525/625 Fall 2023 — D. Ivanšić

Real Numbers

Type A problems (5pts each)

For the sets listed below do the following (justify each answer a little, perhaps with a drawing, but do not go into an axiom-based proof):
a) Find an upper bound and a lower bound, if any.
b) Find $\inf S$ and $\sup S$, if they exist.

A1. $S = \{1, 3, 7, 10\}$ **A2.** $S = (-\infty, 5]$ **A3.** $S = (-1, 4)$ (open interval)

A4. $S = \{1\} \cup [2, 4] \cup \{5\}$ **A5.** $S = \{x \in \mathbf{Q} \mid 3 \leq x < 7\}$

A6. $S = \left\{ \frac{1}{n} \mid n \in \mathbf{N} \right\}$ **A7.** $S = \left\{ (-1)^n \frac{1}{n} \mid n \in \mathbf{N} \right\}$

A8. $S = \{e^x \mid x \in \mathbf{R}\}$ **A9.** $S = \{\arctan x \mid x \in \mathbf{R}\}$

A10. $S = \left\{ (-1)^n \left(2 - \frac{1}{n}\right), \mid n \in \mathbf{N} \right\}$ **A11.** $S = \{x \in \mathbf{Q} \mid x^2 < 2\}$

Type B problems (8pts each)

B1. Show that the set of complex numbers $\mathbf{C}$ cannot be made into an ordered field. In other words, show that it is impossible to find a set $P \subset \mathbf{C}$ that satisfies the order properties from 2.1.5. (Hint: suppose it is possible and derive a contradiction using the element $i \in \mathbf{C}$.)

B2. Formulate and prove statements for $\inf S$ that are analogous to Lemmas 2.3.3 and 2.3.4.

B3. Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$.
a) Show that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.
b) Use a) and induction to prove the binomial formula: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.

B4. Let $f : X \to Y$ be a function and $\{B_n, n \in \mathbf{N}\}$ a collection of subsets of Y. Prove the following: $f^{-1}\left(\bigcup_{n=1}^{\infty} B_n \right) = \bigcup_{n=1}^{\infty} f^{-1}(B_n)$ and $f^{-1}\left(\bigcap_{n=1}^{\infty} B_n \right) = \bigcap_{n=1}^{\infty} f^{-1}(B_n)$.

B5. Show directly that the set X of sequences of 0s and 1s is uncountable. Then, using the characteristic function χ_A, show there is a bijective map between the set X and the set of all subsets of $\mathbf{N}$, which we have shown to be uncountable.

Type C problems (12pts each)

C1. (Juicy!) Do problem 2.1.9 for the set $K = \{s + t\sqrt[3]{2} + u\sqrt[3]{4} \mid s, t, u \in \mathbf{Q}\}$. Or, if you prefer, do it more generally for the set $K = \{s + t\alpha + u\alpha^2 \mid s, t, u \in \mathbf{Q}\}$, where $\alpha \notin \mathbf{Q}$ is a number for which $\alpha^3 \in \mathbf{Q}$. (Hint for part b of 2.1.9: you are trying to rationalize the denominator in $\frac{1}{s+t\alpha+u\alpha^2}$. If you multiply numerator and denominator by $x + y\alpha + z\alpha^2$, what conditions on $x, y, z \in \mathbf{Q}$ give you a rational denominator, and can you always satisfy the conditions?)

C2. Show for every $n \geq 3$: $\sqrt[n+1]{n+1} < \sqrt[n]{n}$. (Hint: show it is equivalent to $(n+1)^n < n^{n+1}$, and prove this, perhaps with the help of the binomial formula.)