4.1 Limits of Functions

Definition 4.1.1. Let $A \subseteq \mathbf{R}$. A point $c \in \mathbf{R}$ is a *cluster point* of A if for every $\delta > 0$ there exists at least one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$. Alternatively, we can say that every δ-neighborhood of c contains a point from A other than c.

Example. Show 0, 1, 2 are all cluster points of the sets $[0, 2]$ and $(0, 2)$. Identify all the cluster points of these sets.

Example. $\sqrt{2}$ is a cluster point of $\mathbf{Q}$.

Theorem 4.1.2. A number c is a cluster point of $A \subseteq \mathbf{R}$ if and only if there exists a sequence (a_n) in A such that $\lim a_n = c$ and $a_n \neq c$ for all $n \in \mathbf{N}$.

Proof.

Note. A finite set has no cluster points.

Definition 4.1.4. Let $A \subseteq \mathbf{R}$, c a cluster point of A, and let $f : A \to \mathbf{R}$ be a function. We say that a real number L *is the limit of f at c* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$ and $x \in A$, then $|f(x) - L| < \varepsilon$. We write

$$\lim_{x \to c} f(x) = L \text{ or } f(x) \to L \text{ as } x \to c$$

and say that f *converges to* L at c. If the limit of f at c does not exist, we say that f *diverges* at c.

The number δ depends on ε, so it is sometimes written as $\delta(\varepsilon)$ (but we do not define a function $\varepsilon \mapsto \delta(\varepsilon)$). Typically, the smaller ε is, the smaller the corresponding $\delta(\varepsilon)$.

Note. The definition is equivalent to: for every ε-neighborhood $V_\varepsilon(L)$ of L, there is a δ-neighborhood of c such that every $x \in V_\delta(c) \cap A$ is sent to $f(x) \in V_\varepsilon(L)$, that is $f(V_\delta(c)) \subseteq V_\varepsilon(L)$. This language is used to generalize the idea of a limit to topological spaces.

Theorem 4.1.5. Let c be a cluster point of A and $f : A \to \mathbf{R}$. If f has a limit at c, it is unique.

Proof.

Example. $\lim_{x \to c} b = b$ and $\lim_{x \to c} x = c$

Sequential Criterion 4.1.8. Let c be a cluster point of A and $f : A \to \mathbf{R}$. The following are equivalent.

1) $\lim_{x \to c} f(x) = L$
2) For every sequence (x_n) in A such that x_n converges to c and $x_n \neq c$ for all $n \in \mathbf{N}$, the sequence $f(x_n)$ converges to L.

Proof.

Divergence Criteria 4.1.9. Let c be a cluster point of A and $f : A \to \mathbf{R}$.

a) f does not have limit L at c if and only if there exists a sequence (x_n) in A such that $x_n \neq c$ for all $n \in \mathbf{N}$ and $x_n \to c$ but $f(x_n)$ does not converge to L.
b) f does not have a limit at c if and only if there exists a sequence (x_n) in A such that $x_n \neq c$ for all $n \in \mathbf{N}$ and $x_n \to c$ but $f(x_n)$ does not converge.

Proof. a) is the negation of the equivalence 1) $\Longleftrightarrow$ 2) from above;
b$\Longleftarrow$) The assumption is "for every L, $\lim f(x_n) \neq L$," which by $\neg 2) \Longrightarrow \neg 1)$ implies for every L, $\lim_{x \to c} f(x) \neq L$; b$\Longrightarrow$) prove the contrapositive: assume for every sequence (x_n), $\lim f(x_n)$ exists. Then for every two sequences (x_n) and (y_n), $\lim f(x_n) = \lim f(y_n)$, because the limit for the shuffled sequence exists. Now 2) $\Longrightarrow$ 1) gives us that $\lim_{x \to c} f(x)$ exists.

Example. $\lim_{x \to 0} \frac{1}{x}$ does not exist.

Example. $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

4.2 Limit Theorems

Definition 4.2.1. Let c be a cluster point of A and $f : A \to \mathbf{R}$. We say f is *bounded on a neighborhood of* c if there exists a δ-neighborhood of c and an $M > 0$ such that $|f(x)| < M$ for all $x \in A \cap V_\delta(c)$.

Theorem 4.2.2. Let c be a cluster point of A and $f : A \to \mathbf{R}$. If f has a limit at c, then f is bounded on some neighborhood of c.

Proof.

Theorem 4.2.4. Let c be a cluster point of A and $f, g : A \to \mathbf{R}$. Suppose $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then

a) $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$ $\qquad$ $\lim_{x\to c} bf(x) = b \lim_{x\to c} f(x)$

$\lim_{x\to c}(f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$ $\qquad$ $\lim_{x\to c}(f(x)g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

b) If, furthermore, $g(x) \neq 0$ for all $x \in A$ and $\lim_{x\to c} g(x) \neq 0$, then $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$.

Proof. Similar to Theorem 3.2.3.

Useful exercise: redo the rest of the proofs for limits of functions without using the sequential criterion.

Theorem 4.2.6. Let c be a cluster point of A and $f, g, h : A \to \mathbf{R}$. Suppose $\lim_{x \to c} f(x)$, $\lim_{x \to c} g(x)$ and $\lim_{x \to c} h(x)$ exist. If there is a δ such that

$$f(x) \leq g(x) \leq h(x) \text{ for all } x \in A \cap V_\delta(c), \text{ then } \lim_{x \to c} f(x) \leq \lim_{x \to c} g(x) \leq \lim_{x \to c} h(x).$$

Proof. is similar to analogous theorem for sequences.

Squeeze Theorem 4.2.7. Let c be a cluster point of A and $f, g, h : A \to \mathbf{R}$. If there is a δ such that

$$f(x) \leq g(x) \leq h(x) \text{ for all } x \in A \cap V_\delta(c)$$

and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$.

Proof.

Example. For any polynomial $p : \mathbf{R} \to \mathbf{R}$, $\lim_{x \to c} p(x) = p(c)$. For any rational function f and c such that $f(c)$ is defined (i.e. the denominator is not zero at c), $\lim_{x \to c} f(x) = f(c)$.

Example. $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Theorem 4.2.9. Let c be a cluster point of A and $f : A \to \mathbf{R}$. If $\lim_{x \to c} f(x) > 0$ ($\lim_{x \to c} f(x) < 0$), then there exists a δ-neighborhood $V_\delta(c)$ such that $f(x) > 0$ ($f(x) < 0$) for all $x \in A \cap V_\delta(c)$.

Proof.