3.1 Sequences and Their Limits

A sequence is essentially an infinite list of real numbers:

$$1, 2, 4, 8, 16, \dots, 2^n, \dots$$
$$1, -1, 1, -1, 1, \dots, (-1)^{n-1} \dots$$
$$0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \dots, (-1)^{n-1}\frac{n-1}{n}$$

More formally, we have:

Definition 3.1.1. A *sequence* of real numbers is a function $X : \mathbf{N} \to \mathbf{R}$.

Notation: $X(n)$ is usually written as x_n and called the *n-th term* of a sequence.
Notation representing a sequence: X, (x_n), $(x_n \mid n \in \mathbf{N})$, $(x_1, x_2, x_3, \dots)$

Note the difference: $\{x_n \mid n \in \mathbf{N}\}$ = set that contains the terms of (x_n) = range of X
$(x_n \mid n \in \mathbf{N})$ is the sequence, so takes the order into account

For the second example above $\{x_n \mid n \in \mathbf{N}\} =$

Example. $B = (b, b, b, \dots)$ is the constant sequence b.

Example. $(b^n \mid n \in \mathbf{N})$ is the geometric sequence: $x_n = b \cdot x_{n-1}$

Example. $\left(\frac{1}{\sqrt{n}} \mid n \in \mathbf{N}\right)$

Example. Sequences may be given recursively: $f_1 = 1$, $f_2 = 1$, $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. We get $(1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots)$, called the *Fibonacci sequence*.

Definition 3.1.3. A sequence (x_n) *converges to* $x \in \mathbf{R}$ if for every $\varepsilon > 0$ there exists a $K \in \mathbf{N}$ such that for all $n \geq K$, $|x_n - x| < \varepsilon$. We also say (x_n) *has a limit*, it is x.

Note. The number K depends on ε, so it is sometimes written as $K(\varepsilon)$ (but we do not define a function $\varepsilon \mapsto K(\varepsilon)$). Typically, the smaller ε is, the greater the corresponding $K(\varepsilon)$.

If a sequence has a limit, it is called *convergent*, otherwise, it is *divergent*.
Notation: $x = \lim x_n$ or $x = \lim X$ or $x_n \to x$.

Note. The definition is equivalent to: $x = \lim x_n$ if and only if for every ε-neighborhood of x, *all but finitely many terms of* x are in $V_\varepsilon(x)$.

Example. $\lim \dfrac{1}{n} = 0$.

Example. For the constant sequence $(b) = (b, b, b, \dots)$, $\lim b = b$.

Example. The sequence $((-1)^{n-1} n \in \mathbf{N}) = (1, -1, 1, -1, 1, \dots,)$ is divergent.

Example. For every $c > 1$, $\lim \dfrac{1}{n^c} = 0$.

Theorem 3.1.4. A sequence in $\mathbf{R}$ can have at most one limit.

Proof.

Definition 3.1.8. Let $X = (x_1, x_2, \dots)$ be a sequence. The *m-tail* X_m of the sequence X is the sequence we get from X by deleting the first m terms:

$$X_m = (x_{m+1}, x_{m+2}, \dots)$$

Example. If $X = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, \dots\right)$, then $X_4 =$

Theorem 3.1.9. Let X be a sequence, $m \in \mathbf{N}$. Then X converges if and only if X_m converges. In this case, $\lim X = \lim X_m$.

Another way to state the gist of the theorem: convergence does not depend on what happens in the first finitely many terms.

Example. The sequence $\left(1, 10, 100, \dots, 10^{57}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right)$ converges.

Definition. A sequence X has a property *ultimately* if some tail of X has this property.

Example. The sequence $(1, 2, 3, \dots, 101, 101, 101, \dots)$ is ultimately constant.

Note. A sequence X converges to x if and only if for every $\varepsilon > 0$, all the terms of X are ultimately in the ε-neighborhood of x.

Theorem 3.1.10. Let (x_n) be a sequence, $x \in \mathbf{R}$ and let (a_n) be a sequence that converges to 0 with $a_n \geq 0$ for all $n \in \mathbf{N}$. If for some constant C and some $m \in \mathbf{N}$ we have

$$|x_n - x| \leq C a_n, \text{ for all } n \geq m.$$

Then $\lim x_n = x$.

Proof.

Example. If $a > 0$, then $\lim \dfrac{1}{1 + na} = 0$.

Example. If $|b| < 1$, then $\lim b^n = 0$.

Example. For every $c > 0$, $\lim \sqrt[n]{c} = 1$.

Example. $\lim \sqrt[n]{n} = 1$.

3.2 Limit Theorems

Definition 3.2.1. A sequence $X = (x_n)$ is *bounded* if there exists a number $M > 0$ such that $|x_n| < M$ for all $n \in \mathbf{N}$.

Note. A sequence is bounded if and only if $\{x_n \mid n \in \mathbf{N}\}$ is a bounded set.

Theorem 3.2.2. A convergent sequence is bounded.

Proof.

Given sequences X and Y we can form sequences $X \pm Y$, $X \cdot Y$, cX and $\dfrac{X}{Y}$ in the obvious way.

$X = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right)$ $Y = \left(1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^{n-1}}, \dots\right)$

$X \pm Y =$ $X \cdot Y =$

$5X =$ $\dfrac{X}{Y} =$

Note. The sequence $\dfrac{X}{Y}$ is defined only if $y_n \neq 0$ for all $n \in \mathbf{N}$.

Theorem 3.2.3. Let X and Y converge to x and y, respectively. Then

a) $\lim(X+Y) = x + y = \lim X + \lim Y$ $\lim cX = cx = c \lim X$

$\lim(X-Y) = x - y = \lim X \lim Y$ $\lim(X \cdot Y) = x \cdot y = \lim X \cdot \lim Y$

b) If, furthermore, $y_n \neq 0$ for all $n \in \mathbf{N}$, then $\lim \dfrac{X}{Y} = \dfrac{x}{y} = \dfrac{\lim X}{\lim Y}$.

Proof. a)

b)

Example. If $f(x)$ is a rational function, then $\lim f(x_n) = f(\lim x_n)$.

Example. $\lim \dfrac{4n^3 - 5n^2 + 4n + 7}{n^3} =$

Example. $\lim \dfrac{-2n^4 + n^2 - 11n + 5}{n^5 + n^4 + 2n^3 + 1} =$

Example. $\lim \dfrac{n^3 - 4n^2 + 10}{3n^3 + n^2 + 2n - 3} =$

Theorem 3.2.4. If $x = \lim x_n$, and for some $m \in \mathbf{N}$, $x_n \geq 0$ for all $n \geq m$, then $x \geq 0$.

Proof.

Theorem 3.2.5. If (x_n) and (y_n) are convergent, and for some $m \in \mathbf{N}$, $x_n \leq y_n$ for all $n \geq m$, then $\lim x_n \leq \lim y_n$.

Proof.

Theorem 3.2.6. If (x_n) is convergent, and for some $m \in \mathbf{N}$, $a \leq x_n \leq b$ for all $n \geq m$, then $a \leq \lim x_n \leq b$.

Proof.

Squeeze Theorem 3.2.7. Let (x_n), (y_n) and (z_n) be sequences such that

$$\text{for some } m \in \mathbf{N}, x_n \leq y_n \leq z_n \text{ for all } n \geq m, \text{ and } \lim x_n = \lim z_n$$

Then (y_n) is convergent, and $\lim y_n = \lim x_n = \lim z_n$.

Proof.

Example. Show that $\lim \dfrac{e^{\frac{1}{n}} \cos \frac{1}{n} - e^{-n} \sin(n^2 + n)}{n} = 0$.

Theorem 3.2.9. If sequence (x_n) converges to x, then the sequence $(|x_n|)$ converges to $|x|$.

Theorem 3.2.10. If sequence (x_n) converges to x, and $x_n \geq 0$ for all $n \in \mathbf{N}$, then the sequence $(\sqrt{x_n})$ converges to $\sqrt{x}$.

Proof. Read in book.

Theorem 3.2.11 (ratio test). Let (x_n) be a sequence such that $x_n > 0$ for all $n \in \mathbf{N}$, and $\lim \dfrac{x_{n+1}}{x_n} = L$. If $L < 1$, then (x_n) converges and $\lim x_n = 0$.

Proof.

Example. Show that $\lim \dfrac{(n!)^2}{(2n)!} = 0$.

3.3 Monotone Sequences

Definition 3.3.1. A sequence (x_n) is *increasing* if $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \dots$
decreasing if $x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \geq \dots$

A sequence is *monotone* if it is increasing or decreasing.

Example. These sequences are monotone.

$(1, 2, 3, 4, \dots)$ $\qquad$ $(1, a, a^2, a^3, \dots)$ for $a > 1$

$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$ $\qquad$ $(1, b, b^2, b^3, \dots)$ for $0 < b < 1$

Example. These sequences are not monotone.

$(1, 2, 3, 6, 5, 4, 7, 8, 9, 12, 11, 10, \dots)$ $\qquad$ $(1, 2, 1, 2, \dots)$

Theorem 3.3.2. A monotone sequence is convergent if and only if it is bounded.
Furthermore,

a) If (x_n) is increasing and bounded, then $\lim x_n = \sup\{x_n \mid n \in \mathbf{N}\}$.
b) If (y_n) is decreasing and bounded, then $\lim y_n = \inf\{y_n \mid n \in \mathbf{N}\}$

Proof.

Example. Consider the sequence given by $x_1 = 0$, $x_{n+1} = \frac{1}{5}(3x_n + 1)$. Is it monotone or bounded? If both, what is its limit?

Now consider the sequence given by $x_1 = 2$, $x_{n+1} = \frac{1}{5}(3x_n + 1)$. Is it monotone or bounded? If both, what is its limit?

For the sequence $x_1 = b$, $x_{n+1} = \frac{1}{5}(3x_n + 1)$, what condition determines if it is increasing or decreasing?

Write the overall conclusion on the convergence of the sequence $x_1 = b$, $x_{n+1} = \dfrac{1}{5}(3x_n + 1)$.

Example. Read example 3.3.5 in book, for any $a \geq 0$, a sequence that converges to $\sqrt{a}$.

Example. Show that the sequence $h_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$ is not bounded.

For which n is $h_n > 51$?

Example. Show that the seqence $e_n = \left(1 + \dfrac{1}{n}\right)^n$ is increasing and bounded.

By the Monotone Convergence Theorem, the sequence (e_n) converges to some positive real number which is called e.

Fact. e is irrational and *transcendental*, which means it is not a solution of any equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ with rational coefficients $a_0, \ldots, a_n$.

3.4 Subsequences

Definition 3.4.1. Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < \dots$ be a strictly increasing sequence of natural numbers. The sequence $X' = (x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots)$ is called a *subsequence* of X.

Example. If $X = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$, here are some subsequences:

$\left(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\right)$

$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \frac{1}{17} \dots\right)$

$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \dots\right)$

any tail of X

Not a subsequence: $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{6}, \frac{1}{12}, \frac{1}{10}, \dots\right)$

Theorem 3.4.2. If (x_n) converges to x, then every subsequence of (x_n) converges to x.

Proof.

Example. We can show $\lim \sqrt[n]{c} = 1$ and $\lim \sqrt[n]{n} = 1$ with a subsequence trick.

Theorem 3.4.4. For a sequence (x_n), the following are equivalent:

1) (x_n) does not converge to $x \in \mathbf{R}$.
2) There exists an ε_0 such that for every $k \in \mathbf{N}$ there is an $n_k \in \mathbf{N}$ so that $n_k \geq k$ and $|x_{n_k} - x| > \varepsilon_0$.
3) There exists an ε_0 and a subsequence x_{n_k} such that $|x_{n_k} - x| > \varepsilon_0$.

A Divergence Criterion 3.4.5. If 1) or 2) holds, the sequence (x_n) is divergent.

1) (x_n) has two convergent subsequences (x_{n_k}) and (x_{r_k}) whose limits are not equal.
2) (x_n) is unbounded.

Proof.

Monotone Subsequence Theorem 3.4.7. Every sequence (x_n) has a monotone subsequence.

Proof.

Bolzano-Weierstrass Theorem 3.4.8. A bounded sequence has a convergent subsequence.

Proof.

Theorem 3.4.9. Let (x_n) be a bounded sequence with the property that every subsequence converges to the same real number x. Then (x_n) converges to x.

Proof.

3.5 The Cauchy Criterion

Definition 3.5.1. A sequence $X = (x_n)$ of real numbers is called a *Cauchy sequence* if for every $\varepsilon > 0$ there exists a $K \in \mathbf{N}$ such that for all $n, m \geq K$ we have $|x_n - x_m| < \varepsilon$.

Example. The sequence $((-1)^n)$ is not Cauchy.

Example. The sequence $f_n = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \cdots + \dfrac{1}{n!}$ is a Cauchy sequence.

Lemma 3.5.3. If (x_n) is a convergent sequence, then it is a Cauchy sequence.

Proof.

Lemma 3.5.4. A Cauchy sequence is bounded.

Proof.

Cauchy Convergence Criterion 3.5.5. A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Proof.

Example. The sequence (f_n), defined earlier, converges. It converges to e because it can be shown that $|f_n - e_n| < \frac{6}{n}$, where e_n is the sequence from section 3.3. One can use the sequence f_n as the sequence that defines e.

Note. The convergence criterion is not true in $\mathbf{Q}$: a Cauchy sequence of rationals need not converge to a rational number.

Consider example 3.3.5 in book: $s_1 = 1$, $s_{n+1} = \frac{1}{2}\left(s_n + \frac{2}{s_n}\right)$ is a Cauchy sequence of rational numbers, because it converges in $\mathbf{R}$ (but it can be directly proved, too). Furthermore, it converges to a number whose square is 2, which cannot be in $\mathbf{Q}$.

Nonconvergence in $\mathbf{Q}$ of a Cauchy sequence is because $\mathbf{Q}$ does not have the completeness property of $\mathbf{R}$, which crept into the Convergence Criterion via the Monotone Convergence Theorem, and that one uses completeness in an unavoidable way.

Actually, one can show that every Cauchy sequence converges if and only if every set bounded above has a supremum. This may be used to expand the idea of completeness to sets that do not have order properties. For such a set, we can define completeness as "every Cauchy sequence converges."

Example. The set $(\mathbf{C}, +, \cdot)$ is complete by the Cauchy sequence definition. Note that $|\ |$ is defined in $\mathbf{C}$, so the definition of convergence and Cauchy sequence makes sense.

Example. Cauchy sequences can be used to construct $\mathbf{R}$ rigorously. Consider the set of all Cauchy sequences with terms in $\mathbf{Q}$:

$$R = \{(x_n) \mid x_n \in \mathbf{Q} \text{ for all } n \in \mathbf{N} \text{ and } (x_n) \text{ is Cauchy}\}$$

Define a relation $\sim$ on R: $(x_n) \sim (y_n)$ if for every $m \in \mathbf{N}$ there is a $K \in \mathbf{N}$ such that $|x_n - y_n| < \frac{1}{m}$ for all $n \geq K$. This is meant to capture all Cauchy sequences that "converge" to the same limit in a class. One shows:

1) The relation $\sim$ is an equivalence relation (reflexive, commutative and transitive), which breaks up R into a set of classes $R/\sim$.
2) On the set $R/\sim$ we can define the operations $+, \cdot$ satisfying algebraic properties of $\mathbf{R}$.
3) In the set R we can define a set P of "positive elements:" $(x_n) \in P$ if there exists an $m \in \mathbf{N}$ such that there is a $K \in \mathbf{N}$ so $x_n > \frac{1}{m}$ for all $n \geq K$. It can be shown that if $(x_n) \sim (y_n)$ and $(x_n) \in P$, then $(y_n) \in P$, which yields a set $P/\sim$ in $R/\sim$, "positive elements" of $R/\sim$. One shows that $P/\sim$ satisfies all the order properties of $\mathbf{R}$, so we can introduce the relation $<$.
4) Using $<$, we can define boundedness of a set and a supremum in $R/\sim$, and then show $R/\sim$ satisfies the completeness property of $\mathbf{R}$.

One then defines the set of real numbers as $\mathbf{R} = R/\sim$.

Definition 3.5.7. We say a sequence (x_n) is *contractive* if there exists a constant $C \in (0,1)$ such that for all $n \in \mathbf{N}$

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$$

Theorem 3.5.8. Every contractive sequence is a Cauchy sequence, hence convergent.

Proof.

Corollary 3.5.9. If (x_n) is a contractive sequence with constant $C \in (0,1)$ and limit x, then

$$i) \quad |x - x_n| \leq \frac{C^{n-1}}{1-C}|x_2 - x_1| \qquad ii) \quad |x - x_n| \leq \frac{C}{1-C}|x_n - x_{n-1}|$$

Example. Consider the sequence $x_1 = a$, $x_{n+1} = \frac{1}{5}(3x_n + 1)$. Show that the sequence is contractive and verify estimates in the corollary.

3.6 Properly Divergent Sequences

We define precisely what it means for a sequence to tend to ∞ or $-\infty$.

Definition 3.6.1. A sequence (x_n) of real numbers

1) *tends to* ∞, if for every $M \in \mathbf{R}$ there exists a $K \in \mathbf{N}$ such that for all $n \geq K$, $x_n > M$. We write $\lim x_n = \infty$.
2) *tends to* $-\infty$, if for every $M \in \mathbf{R}$ there exists a $K \in \mathbf{N}$ such that for all $n \geq K$, $x_n < M$. We write $\lim x_n = -\infty$.

In either case, we say (x_n) is *properly divergent.*

Example. $\lim n = \infty$ and $\lim n^c = \infty$ for every $c > 0$.

Example. $\lim c^n = \infty$ for every $c > 1$.

Theorem 3.6.3. A monotone sequence is properly divergent if and only if it is unbounded.

a) If (x_n) is increasing and unbounded, then $\lim x_n = \infty$.
b) If (x_n) is decreasing and unbounded, then $\lim x_n = -\infty$.

Proof.

Theorem 3.6.4. Let (x_n) and (y_n) be sequences so that there exists an $m \in \mathbf{N}$ so $x_n \leq y_n$ for all $n \geq m$.

a) If $\lim x_n = \infty$, then $\lim y_n = \infty$

b) If $\lim y_n = -\infty$, then $\lim x_n = -\infty$

Proof.

Note. Knowing that $\lim y_n = \infty$ does not give you anything about $\lim x_n$.

Theorem 3.6.5. Let (x_n) and (y_n) be sequences of positive real numbers, and suppose that

$$\lim \frac{x_n}{y_n} = L, \text{ for some } L > 0.$$

Then $\lim x_n = \infty$ if and only if $\lim y_n = \infty$.

Proof.

Extended Limit Laws. These give some information on convergence or proper divergence of a sequence built from sequences using algebraic operations $+, -, \cdot, \div$.

$$\frac{1}{0\pm} = \pm\infty \qquad \frac{L}{\pm\infty} = 0 \qquad L \cdot \infty = \begin{cases} \infty & \text{if } L > 0 \\ -\infty & \text{if } L < 0 \end{cases} \qquad \begin{array}{l} \infty + \infty = \infty \\ \infty \cdot \infty = \infty \end{array} \qquad \begin{array}{l} L + \infty = \infty \\ L - \infty = -\infty \end{array}$$

Note that each of the statements is shorthand for a statement about limits. For example,

a) $\frac{1}{0+} = \infty$ stands for: if $\lim x_n = 0$ and $x_n > 0$, then $\lim \frac{1}{x_n} = \infty$.

b) $L \cdot \infty = \infty$ for $L > 0$ stands for: if $\lim x_n = L > 0$ and $\lim y_n = \infty$, then $\lim x_n y_n = \infty$.