2.1 The Algebraic and Order Properties of R

In our mathematical education, we learned numbers starting with natural numbers and then progressed to integers and rational numbers. The latter concepts are essentially based on natural numbers. At the same time, we associated numbers with points on a line, and at some point it became clear that rational numbers are not enough to describe *all* points on the line. We then accept that there is a larger set of numbers associated to points on the line that we call *real* numbers.

One can construct real numbers from rational numbers in a rigorous way, but this is fairly theoretical and takes away from our study of calculus. This is why we skip it and take the existence of real numbers, modeled with a line, on faith (i.e. as axioms).

There exists a set $\mathbf{R}$, called *the set of real numbers* with the following properties.

Algebraic Properties of R 2.1.1. The set $\mathbf{R}$ has binary operations $+$ and $\cdot$ (addition and multiplication) which satisfy:

(A1) $a+b=b+a$, for all $a,b\in\mathbf{R}$ — *commutativity of +*

(A2) $(a+b)+c=a+(b+c)$ for all $a,b,c\in\mathbf{R}$ — *associativity of +*

(A3) There exists an element $0\in\mathbf{R}$ such that $0+a=a+0=a$. — *existence of a zero (neutral) element for +*

(A4) For every $a\in\mathbf{R}$ there exists an element denoted $-a$ such that $(-a)+a=a+(-a)=0$. — *existence of an additive inverse (opposite element) for +.*

(M1) $a\cdot b=b\cdot a$, for all $a,b\in\mathbf{R}$ — *commutativity of ·*

(M2) $(a\cdot b)\cdot c=a\cdot(b\cdot c)$ for all $a,b,c\in\mathbf{R}$ — *associativity of ·*

(M3) There exists an element $1\in\mathbf{R}$, $1\neq 0$, such that $1\cdot a=a\cdot 1=a$. — *existence of a unit (neutral) element for ·*

(M4) For every $a\in\mathbf{R}$, $a\neq 0$ there exists an element denoted $\frac{1}{a}$ such that $\frac{1}{a}\cdot a=a\cdot\frac{1}{a}=1$ — *existence of a multiplicative inverse (reciprocal element) for ·*

(D) For all $a,b,c\in\mathbf{R}$, $a\cdot(b+c)=a\cdot b+a\cdot c$ and $(a+b)\cdot c=a\cdot c+b\cdot c$ — *distributivity of · over +*

Note. In abstract algebra, these properties state that $(\mathbf{R},+,\cdot)$ is a *field.*

Theorem 2.1.2. (Uniqueness of 0, 1)

a) If $z,a\in\mathbf{R}$ such that $z+a=a$, then $z=0$.

b) If $u,b\in\mathbf{R}$, $b\neq 0$ such that $u\cdot b=b$, then $u=1$.

c) For all $a\in\mathbf{R}$, $a\cdot 0=0$.

Proof.

Theorem 2.1.3. (Uniqueness of opposites and reciprocals, zero product property)

a) If $b \in \mathbf{R}$ such that $a + b = 0$, then $b = -a$.
   If $b \in \mathbf{R}$ such that $a \cdot b = 1$, then $b = \frac{1}{a}$.

b) If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

Proof.

Note. If we do not stipulate $1 \neq 0$, there is only one element in $\mathbf{R}$.

Note. If 0 is allowed to have a reciprocal, then $0 = 1$.

We define

$$a^0 = 1 \qquad a^n = \overbrace{a \cdot a \cdot \cdots \cdot a}^{n \text{ times}} \qquad \frac{a}{b} = a \cdot \frac{1}{b} \qquad a^{-n} = \frac{1}{a}^n$$

Using algebraic properties of $\mathbf{R}$, it can be shown that all the algebraic rules for working with real numbers are valid. As usual, we omit the multiplication symbol $\cdot$ where not needed.

For any $n \in \mathbf{N}$ we define the element $n \in \mathbf{R}$ as

$$n = \overbrace{1 + 1 + \cdots + 1}^{n \text{ terms}}$$

Then we can consider sets $\mathbf{N}'$, $\mathbf{Z}'$ and $\mathbf{Q}'$ defined as

$$\mathbf{N}' = \{\underbrace{1 + \cdots + 1}_{n \text{ terms}} \mid n \in \mathbf{N}\} \quad \mathbf{Z}' = \mathbf{N}' \cup \{0\} \cup \{-n \mid n \in \mathbf{N}'\} \quad \mathbf{Q}' = \left\{\frac{a}{b} \mid a \in \mathbf{Z}', b \in \mathbf{Z}', b \neq 0\right\}$$

Later order properties imply that the map $\mathbf{N} \to \mathbf{R}$ given by $n \mapsto 1 + \cdots + 1$ is injective, which gives rise to maps $\mathbf{Z} \to \mathbf{R}$ and $\mathbf{Q} \to \mathbf{R}$, also injective. The images of those maps are $\mathbf{N}'$, $\mathbf{Z}'$ and $\mathbf{Q}'$, so we think of $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{Q}$ of being subsets of $\mathbf{R}$ through their stand-ins $\mathbf{N}'$, $\mathbf{Z}'$ and $\mathbf{Q}'$. We refer to elements of $\mathbf{N}'$, $\mathbf{Z}'$ and $\mathbf{Q}'$ as natural numbers, integers and rational numbers (special numbers in $\mathbf{R}$) and stop writing the primes.

Among elements of $\mathbf{N}$, we distinguish even and odd numbers, of forms $2n$ and $2n - 1$ for some natural number n. These have the usual properties that every natural number is even or odd, that none is both, and that for every $n \in \mathbf{N}$, n^2 is even if and only if n is even.

Theorem 2.1.4. There is no rational number r such that $r^2 = 2$.

Proof.

Note. The theorem does not say anything about the existence of such a real number r.

Order Properties of R 2.1.5. There is a nonempty subset $\mathbf{P} \subseteq \mathbf{R}$ that satisfies the following properties:

(i) If $a, b \in \mathbf{P}$, then $a + b \in \mathbf{P}$

(ii) If $a, b \in \mathbf{P}$, then $a \cdot b \in \mathbf{P}$

(iii) For every $a \in \mathbf{R}$, exactly one of the following holds: $a \in \mathbf{P}$, $a = 0$ or $-a \in \mathbf{P}$. *trichotomy property*

Note that by (iii), the set $\mathbf{R}$ is broken up into three disjoint sets:

$\mathbf{P}$, called *positive real numbers* $\quad \{0\} \quad -\mathbf{P} = \{-a \mid a \in \mathbf{P}\}$, called *negative real numbers*

Definition 2.1.6. Let $a, b \in \mathbf{R}$.

a) We write $a > b$ or $b < a$ if $a - b \in \mathbf{P}$

b) We write $a \geq b$ or $b \leq a$ if $a - b \in \mathbf{P} \cup \{0\}$

The trichotomy property then implies that exactly one of the following holds:

$$a < b \qquad a = b \qquad a > b$$

Theorem 2.1.7. (properties of inequalities) Let $a, b, c \in \mathbf{R}$.

a) If $a > b$ and $b > c$, then $a > c$.

b) If $a > b$, then $a + c > b + c$.

c) If $a > b$ and $c > 0$, then $ac > ab$. If $a > b$ and $c < 0$, then $ac < bc$.

Proof.

Note. Theorem 2.1.7 and Order Properties 2.1.5 are equivalent. We could have started with a relation $<$ that satisfies the trichotomy property and conditions of Theorem 2.1.7 and arrived at the set $\mathbf{P}$. After this section, we will mostly forget the set $\mathbf{P}$ and simply work with properties of inequalities, as we are used to.

Theorem 2.1.8.

a) If $a \neq 0$, then $a^2 > 0$.

b) $1 > 0$

c) If $n \in \mathbf{N}$, then $n > 0$.

Proof.

Note. There does not exist a smallest positive number.

Theorem 2.1.9. If $a \in \mathbf{R}$ is such that $0 \leq a < \varepsilon$ for every $\varepsilon > 0$, then $a = 0$.

Proof.

Theorem 2.1.10. Let $a, b \in \mathbf{R}$.

a) If $ab > 0$, then $a > 0$ and $b > 0$, or $a < 0$ and $b < 0$.
b) If $ab < 0$ then $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$.

Proof.

Example. Bernoulli's inequality. Let $x \in \mathbf{R}$ and $n \in \mathbf{N}$, then

$$(1 + x)^n \geq 1 + nx \text{ for } x \geq -1.$$

2.2 Absolute Value

Definition 2.2.1. $|a| = \begin{cases} a, & \text{if } a > 0 \\ 0, & \text{if } a = 0 \\ -a, & \text{if } a < 0 \end{cases}$

Note. Due to trichotomy, $|a| > 0$.

Theorem 2.2.2.

a) $|ab| = |a||b|$
b) $|a|^2 = a^2$
c) For a $c \geq 0$, $|a| \leq c$ if and only if $-c \leq a \leq c$.
d) $-|a| \leq a \leq |a|$.

Proof.

Triangle Inequality 2.2.3. For all $a, b \in \mathbf{R}$, $|a + b| \leq |a| + |b|$.

Proof.

Corollary 2.2.4.

a) $||a| - |b|| \leq |a - b|$

b) $|a - b| \leq |a| + |b|$

Proof.

Note. On the real line, $|a - b| =$ distance from a to b.

Definition 2.2.7. For an $a \in \mathbf{R}$ and $\varepsilon > 0$ we define the *ε-neighborhood of a* as

$$V_\varepsilon(a) = \{x \in \mathbf{R} \mid |x - a| < \varepsilon\}$$

Theorem 2.2.8. For an $a \in \mathbf{R}$, if $x \in V_\varepsilon(a)$ for every $\varepsilon > 0$, then $x = a$.

Proof.

Example. If x, y are in ε-neighborhoods of a, b, respectively, in what neighborhood of $a + b$ is $x + y$? What statement on accuracy can you make from this fact?

2.3 The Completeness Property of R

Example. Do the following subsets of **R** have a *maximal element*?

$\{1, 3, 7, 10\}$ $[-1, 3]$ $[-1, 3)$ $[-1, \infty)$

For $S = [-1, 3)$, even though 3 is not a maximal element, it has a special property:

1) It is bigger than any element of S.
b) It is the "most efficient" such number.

This is the idea behind the *supremum* of a set.

Definition 2.3.1. Let S be a nonempty subset of **R**.

a) S is *bounded above* it there exists a $u \in \mathbf{R}$ such that $s \leq u$ for all $s \in S$.
Every such number u is called an *upper bound of* S.
b) S is *bounded below* it there exists a $w \in \mathbf{R}$ such that $w \leq s$ for all $s \in S$.
Every such number w is called a *lower bound of* S.
c) S is *bounded* if it is bounded above and below. A set is *unbounded* if it is not bounded.

Example. For each of the following subsets of **R** determine if they are bounded above or below, or bounded. If they are, list some upper or lower bounds.

$[-1, \infty)$ $[-1, 3)$ $(-\infty, 4]$ $\mathbf{N}$

Note. If a set has one upper (lower) bound, it has many.

Definition 2.3.2. Let S be a nonempty subset of $\mathbf{R}$.
If S is bounded above, a number u is called a *supremum* of S (or *least upper bound* of S), denoted $\sup S$, if

1) u is an upper bound of S.
2) If v is any upper bound of S, then $u \leq v$.

If S is bounded below, a number w is called an *infimum* of S (or *greatest lower bound* of S), denoted $\inf S$, if

1) w is a lower bound of S.
2) If t is any lower bound of S, then $t \leq w$.

Note. A supremum or infimum of a set is unique.

Example. For each of the following subsets of $\mathbf{R}$, informally determine its supremum.

$\sup[-1, \infty) =$ $\qquad$ $\sup[-1, 3) =$ $\qquad$ $\sup[-1, 3] =$

Lemma 2.3.3. For a nonempty subset S of $\mathbf{R}$, $u = \sup S$ if and only if

1) u is an upper bound of S (i.e. $s \leq u$ for all $s \in S$).
2) If $v < u$, then there exists an $s' \in S$ such that $v < s'$.

Proof.

Example. Show that $\sup[-1, 3) = 3$.

Lemma 2.3.4. For a nonempty subset S of $\mathbf{R}$, $u = \sup S$ if and only if

1) u is an upper bound of S.
2) For every $\varepsilon > 0$, there exists an $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

Proof. This is a rewording of Lemma 2.3.3.

We have seen so far that many sets, for example, $\mathbf{Q}$ or its extensions by roots, satisfy the algebraic and order axioms listed. Now we add the axiom that uniquely determines the set $\mathbf{R}$.

The Completeness Property of R 2.3.6. Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbf{R}$.

Note. The set of rational numbers $\mathbf{Q}$ does not have this property. For example, the set $\{q \in Q \mid q^2 < 2\}$ does not have a supremum in the set $\mathbf{Q}$.

2.4 Applications of the Supremum Property

Example. Let $S \subseteq \mathbf{R}$ be bounded above and $a \in \mathbf{R}$ and let $a + S$ stand for $\{a + s \mid s \in S\}$ Then $\sup(a + S) = a + \sup S$.

Definition 2.4.2. Let $f : D \to \mathbf{R}$ be a function. We say f is *bounded above (below)* if the set $f(D)$ is bounded above (below). We say f is *bounded* if it is bounded above and below, which is equivalent to there being an $M \in \mathbf{R}$ such that $|f(x)| \leq M$ for all $x \in D$. We define $\sup\limits_{x \in D} f(x)$ to be $\sup f(D)$.

Example. Show that, if $f(x) \leq g(x)$ for all $x \in D$, then $\sup\limits_{x \in D} f(x) \leq \sup\limits_{x \in D} g(x)$.

Theorem 2.4.7. There exists a positive real number x such that $x^2 = 2$.

Proof.

Archimedean Property (N is not bounded above) 2.4.3. For every $x \in \mathbf{R}$ there exists a number $n_x \in \mathbf{N}$ such that $x \leq n_x$.

Proof.

Corollary 2.4.5. If $t > 0$, there exists an $n_t \in \mathbf{N}$ such that $\dfrac{1}{n_t} < t$.

Corollary 2.4.4. $\inf \left\{ \frac{1}{n} \mid n \in \mathbf{N} \right\} = 0$

Corollary 2.4.6. For every $y > 0$, there exists an $n_y \in \mathbf{N}$ such that $n_y - 1 \leq y < n_y$.

Density of Q 2.4.8. Let $x, y \in \mathbf{R}$ with $x < y$. Then there exists an $r \in \mathbf{Q}$ such that $x < r < y$.

Proof.

Density of Irrationals 2.4.9. Let $x, y \in \mathbf{R}$ with $x < y$. Then there exists an irrational number z such that $x < z < y$.

Note. The proof of existence of x such that $x^2 = 2$ can be used to show that $\mathbf{Q}$ does not have the completeness property. In the proof, by making $\varepsilon = \frac{1}{n}$, we can work only with rational numbers. The assumption that $x = \sup S$ exists in $\mathbf{Q}$ leads to the same conclusion that $x^2 = 2$, contradicting the fact that there is no such rational x.