1.3 Finite and Infinite Sets

Counting the number of elements in a set is essentially establishing a bijection between the elements in a set and a subset of natural numbers.

Definition 1.3.1. Let S be a set.

a) The empty set $\emptyset$ is said to have 0 elements.
b) S is said to have n *elements* ($n \in \mathbf{N}$) if there is a bijection between the set $\mathbf{N}_n = \{1, 2, \ldots, n\}$ and S.
c) S is said to be *finite* if it is empty or has n elements for some $n \in \mathbf{N}$.
d) S is said to be *infinite* if it is not finite.

Theorem 1.3.2. If S is a finite set, then its number of elements is a unique number in $\mathbf{N}$.

Theorem B.1. Let $m, n \in \mathbf{N}$, $m > n$. There is no injection $f : \mathbf{N}_m \to \mathbf{N}_n$.

Proof.

Proof of Theorem 1.3.2.

Theorem 1.3.3. $\mathbf{N}$ is an infinite set.

Proof.

Theorem 1.3.4.

a) If A has m elements, B has n elements and $A \cap B = \emptyset$, then $A \cup B$ has $m + n$ elements.
b) If A has m elements and $C \subseteq A$ has 1 element, then $A \setminus C$ has $m - 1$ elements.
c) If C is infinite and $B \subseteq C$ is finite, then $C \setminus B$ is infinite.

Proof of a).

Theorem 1.3.5. Let S and T be sets, and let $T \subseteq S$.

a) If S is finite, so is T.
b) If T is infinite, so is S.

Proof.

Definition 1.3.6. Let S be a set.

a) We say S is *denumerable* if there exists a bijection $\mathbf{N} \to S$.
b) We say S is *countable* if S is either finite or denumerable.
c) We say S is *uncountable* if it is not countable.

Example. The sets $E, O \subseteq \mathbf{N}$ of even and odd numbers are denumerable.

Example. The set $\mathbf{Z}$ is denumerable. Note that a bijection with $\mathbf{N}$ can be described by listing the elements of the set in a sequence.

Example. The union of two disjoint denumerable sets is denumerable.

Theorem 1.3.8. The set $\mathbf{N} \times \mathbf{N}$ is denumerable.

Proof.

Theorem 1.3.9. Let S and T be sets, and let $T \subseteq S$.

a) If S is countable, so is T.

b) If T is uncountable, so is S.

Proof.

Theorem 1.3.10. The following are equivalent.

a) S is countable.

b) There exists a surjection $\mathbf{N} \to S$.

c) There exists an injection $S \to \mathbf{N}$.

Proof.

Theorem 1.3.11. $\mathbf{Q}$ is denumerable.

Proof.

Theorem 1.3.12. If A_m is a countable set for each $m \in \mathbf{N}$, then $\bigcup_{m=1}^{\infty} A_m$ is countable.

Proof.

Cantor's Theorem 1.3.13. For any set A, there is no surjection $A \to \mathcal{P}(A)$, where $\mathcal{P}(A)$ is the set of all subsets of A.

Proof.

Note. Cantor's Theorem means that $\mathcal{P}(\mathbf{N})$ is not denumerable, since there is no surjection $N \to \mathcal{P}(\mathbf{N})$. Therefore, $\mathcal{P}(\mathbf{N})$ is uncountable.

Example. Show that the set $S = \{A \subseteq \mathbf{N} \mid A \text{ is infinite}\}$ of all infinite subsets of $\mathbf{N}$ is uncountable.