| **Calculus 1 — Exam 1**<br>**MAT 250, Fall 2023 — D. Ivanšić** | **Name:** ______________________ |
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| | *Show all your work!* |

**1.** (16pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

$\lim_{x\to -4+} f(x) =$

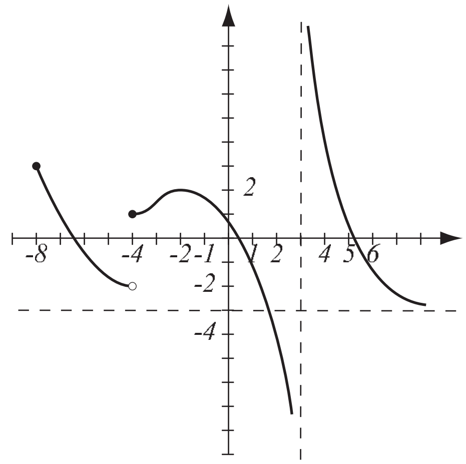
$\lim_{x\to -4-} f(x) =$

$\lim_{x\to -4} f(x) =$

$\lim_{x\to \infty} f(x) =$

$\lim_{x\to 3-} f(x) =$

$\lim_{x\to 3+} f(x) =$




List points in $[-8,\infty)$ where $f$ is not continuous and justify why it is not continuous at those points.

**2.** (6pts) Let $\lim_{x\to 1} f(x) = 4$ and $\lim_{x\to 1} g(x) = -3$. Use limit laws to find the limit below and show each step.

$$\lim_{x\to 1} \frac{x^3 + g(x)^2}{2+\sqrt{f(x)}} =$$

**3.** (10pts) Find $\lim_{x\to 0+} \sqrt{x}\left(3+\sin\frac{1}{x}\right)$. Use the theorem that rhymes with a vegetable that looks like small green balls.

Find the following limits algebraically. Do not use the calculator.

**4.** (7pts) $\lim_{x\to\infty} \dfrac{x^2 - 4x + 2}{2x + 3} =$

**5.** (5pts) $\lim_{x\to 1} \dfrac{x - 1}{x^2 + 5x - 6} =$

**6.** (7pts) $\lim_{x\to 4} \dfrac{\sqrt{x} - 2}{x - 4} =$

**7.** (6pts) $\lim_{x\to 4^+} \dfrac{x - 1}{4 - x} =$

**8.** (7pts) $\lim_{x\to 0} \dfrac{\sin(4x)\tan x}{x^2} =$

**9.** (14pts) The equation $x^3 - 7 = \sqrt{x}$ is given.
a) Use the Intermediate Value Theorem to show it has a solution in the interval $(0, 3)$.
b) Use your calculator to find an interval of length at most 0.01 that contains a solution of the equation. Then use the Intermediate Value Theorem to justify why your interval contains the solution.

**10.** (10pts) Consider the limit $\lim\limits_{x \to 2} \dfrac{\sqrt{x} - \sqrt{2}}{x - 2}$. Use your calculator (don't forget parentheses) to estimate this limit with accuracy 3 decimal points. Write a table of values (no more than 5 per table) that will support your answer.

| $x$ | $\dfrac{\sqrt{x}-\sqrt{2}}{x-2}$ |
|---|---|
| | |

| $x$ | $\dfrac{\sqrt{x}-\sqrt{2}}{x-2}$ |
|---|---|
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**11.** (12pts) Consider the function defined below.
a) Explain why the function is continuous on intervals $(0, 2)$ and $(2, \infty)$
b) For which $c$ is the function continuous at point $x = 2$?

$$f(x) = \begin{cases} x^2 - cx, & \text{if } 0 < x < 2 \\ \dfrac{c}{x} + 5, & \text{if } x \geq 2. \end{cases}$$

**Bonus.** (10pts) Find the limit algebraically. Do not use the calculator.

$$\lim_{h \to 0} \frac{(2+h)^4 - 16}{h} =$$

| **Calculus 1 — Exam 2** <br> **MAT 250, Fall 2023 — D. Ivanšić** |
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**Name:** ____________________

*Show all your work!*

Differentiate and simplify where appropriate:

**1.** (6pts) $\dfrac{d}{dx}\left(4x^7 - \dfrac{5}{x^6} + \sqrt[3]{x^5} + c^{\frac{3}{2}}\right) =$

**2.** (6pts) $\dfrac{d}{dx}\dfrac{\sqrt{x}}{x^2+1} =$

**3.** (6pts) $\dfrac{d}{du}(u+2)^4(u-1)^3 =$

**4.** (5pts) $\dfrac{d}{d\theta}\dfrac{1-\sin\theta}{1+\sin\theta} =$

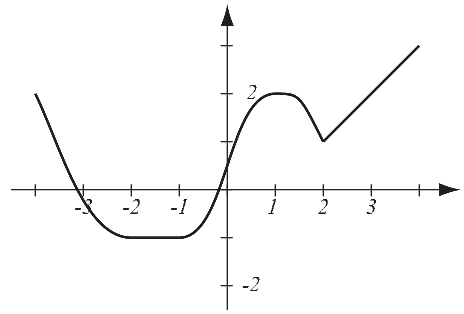
**5.** (6pts) $\dfrac{d}{dx}\sqrt{x+\sqrt{\tan x}} =$

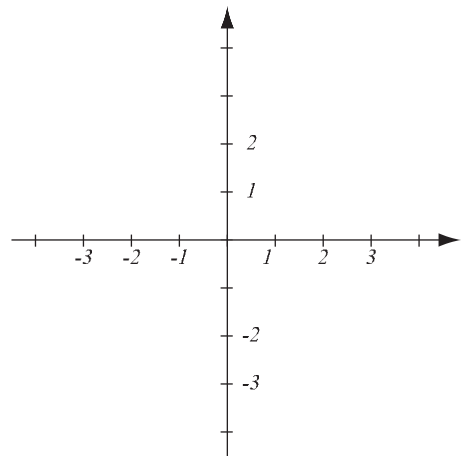
**6.** (7pts) The limit at right represents a derivative $f'(a)$.
a) State $f$ and $a$.
b) To find the limit, evaluate $f'(a)$ using differentiation rules.

$$\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{4}+h\right)-\frac{\sqrt{2}}{2}}{h}$$

**7.** (10pts) The graph of the function $f(x)$ is shown at right.

a) Where is $f(x)$ not differentiable? Why?
b) Use the graph of $f(x)$ to draw an accurate graph of $f'(x)$.




**8.** (12pts) Let $f(x) = \dfrac{2}{x}$.
a) Use the limit definition of the derivative to find the derivative of the function.
b) Check your answer by taking the derivative of $f$ using differentiation rules.
c) Write the equation of the tangent line to the curve $y = f(x)$ at point $(2, 1)$.

**9.** (10pts) Let $g(x) = f(x)^2$ and $h(x) = \dfrac{f(x^2)}{x}$.

a) Find the general expressions for $g'(x)$ and $h'(x)$.
b) Use the table of values at right to find $g'(1)$ and $h'(2)$.

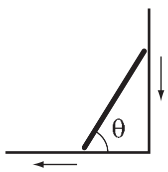
| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 2 | -3 | -1 | 3 |
| $f'(x)$ | -2 | 5 | -4 | 1 |

**10.** (7pts) An arrow shot upwards has position given by the formula $s(t) = -5t^2 + 50t$.
a) Write the formula for the velocity of the arrow at time $t$.
b) What is the velocity of the arrow when it is at height 120 meters on the way up? On the way down?

**11.** (11pts) Use implicit differentiation to find $y'$ in general, and then at point $\left(0, \dfrac{1}{2}\right)$ in particular.

$x^2 + y^2 = 4x^4 + 4y^4 + 8x^2y^2$

**12.** (14pts) A 4-meter ladder is sliding down the wall against which it is leaning. When the bottom of the ladder is 1 meter from the base of the wall, it is moving away from the wall at speed $\frac{1}{5}$ meters per second. How fast is the angle $\theta$ between the ladder and the floor changing at that moment?




**Bonus.** (10pts) Consider the sliding ladder from above. Show that, as the ladder slides down, the top of the ladder has to leave the wall before it hits the ground by doing this:
1) Suppose the top of the ladder stays on the wall. If $x$ and $y$ are distances from the bottom and top of the ladder to the base of the wall, respectively, find the general formula for $y'$ in terms of $x$ and $x'$ (angle does not play a part here).
2) Assuming bottom of ladder slides at a constant $\frac{1}{5}$ meters per second away from the wall, What happens to $y'$ as $x \to 4-$? Is this possible in reality?

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| **Calculus 1 — Exam 3** |
| **MAT 250, Fall 2023 — D. Ivanšić** |

**Name:** ______________________________

*Show all your work!*

Differentiate and simplify where appropriate:

**1.** (4pts) $\dfrac{d}{dx}\, 3^{\tan x} =$

**2.** (6pts) $\dfrac{d}{du}\,(u^2 - 2u + 2)e^u =$

**3.** (7pts) $\dfrac{d}{du}\,\dfrac{e^u + u}{e^u - u} =$

**4.** (7pts) $\dfrac{d}{dx}\,\ln\left(\sin^2 x \cos^2 x\right) =$

**5.** (7pts) $\dfrac{d}{dt}\,\left(t \arccos t - \sqrt{1 - t^2}\right) =$

**6.** (9pts) Use logarithmic differentiation to find the derivative of $y = x^{\arctan x}$.

Find the limits algebraically. Graphs of basic functions will help, as will L'Hospital's rule, where appropriate.

**7.** (2pts) $\lim_{x \to 0+} \ln(2x) =$

**8.** (6pts) $\lim_{x \to \infty} e^{-\frac{x^2+1}{x+3}} =$

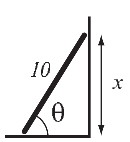
**9.** (7pts) $\lim_{x \to 0} \frac{\cos(4x) - 1}{x^2} =$

**10.** (9pts) $\lim_{x \to 0+} x(\ln x)^2 =$

**11.** (8pts) $\lim_{x \to 0+} (1 - 3x)^{\frac{1}{x}} =$

**12.** (11pts) Let $f(x) = \sqrt{x}$.
a) Write the linearization of $f(x)$ at $a = 4$.
b) Use the linearization to estimate $\sqrt{4.5}$ .
c) In the same coordinate system, draw graphs of the function and the linearization and determine if the estimate in b) is an overestimate or underestimate of $\sqrt{4.5}$ .

**13.** (10pts) A 10-foot ladder leans against the wall. Aiming to compute the angle $\theta$ that the ladder subtends with the floor, we measure the distance $x$ from the floor to the top of the ladder and find it to be 8 feet with maximum error in measurement $\frac{1}{2}$ inch. Use differentials to estimate the maximum possible error when computing the angle $\theta$. (Since you need to express angle $\theta$ as a function of $x$, an inverse trigonometric function is involved.)



**14.** (7pts) Let $f(x) = 2^x + x$. Use the theorem on derivatives of inverses to find $(f^{-1})'(11)$.

**Bonus.** (10pts) Find the derivative and simplify until the bitter end. You will get the derivative of a simpler function. Which one?

$$\frac{d}{dx} \arctan \sqrt{\frac{1-x}{1+x}} =$$

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| **Calculus 1 — Exam 4** |
| **MAT 250, Fall 2023 — D. Ivanšić** |

**Name:**____________________________

*Show all your work!*

**1.** (32pts) Let $f(x) = \ln(x^2 - 2x + 5)$. The domain of this function is all real numbers (you do not have to verify this). Draw an accurate graph of $f$ by following the guidelines.

a) Find the intervals of increase and decrease, and local extremes.

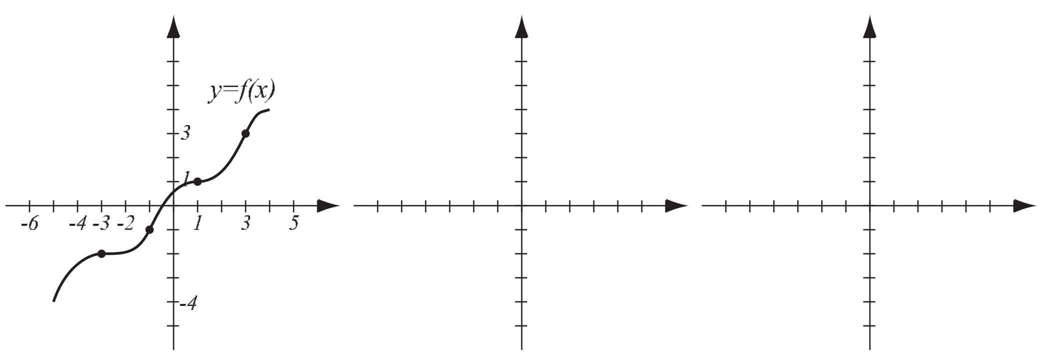
b) Find the intervals of concavity and points of inflection.

c) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

d) Use information from a)–c) to sketch the graph.

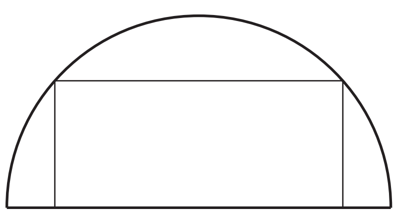
**2.** (18pts) Let $f(x) = \sqrt{2}\sin^2\theta + \frac{4}{3}\cos^3\theta$. Find the absolute minimum and maximum values of $f$ on the interval $[0, \pi]$.

**3.** (14pts) The graph of $f$ is given. Use it to draw the graphs of $f'$ and $f''$ in the coordinate systems provided. Pay attention to increasingness, decreasingness and concavity of $f$. The relevant special points have been highlighted.



**4.** (14pts) Consider $f(x) = \dfrac{1}{x+1}$ on the interval $[0, 2]$.
a) Verify that the function satisfies the assumptions of the Mean Value Theorem.
b) Find all numbers $c$ that satisfy the conclusion of the Mean Value Theorem.

**5.** (22pts) A half-circle of radius 1 meter is given. Among all rectangles that have one side on the diameter and the other two vertices on the half-circle, find the one with the greatest area.



**Bonus.** (10pts) Let $0 < b < 1$.
a) Among all points on the unit circle, mark the point that you think is closest to point $(0, b)$.
b) Using calculus, show that this point is indeed the closest one to point $(0, b)$.

| **Calculus 1 — Exam 5** |
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| **MAT 250, Fall 2023 — D. Ivanšić** |

**Name:** ______________________________

*Show all your work!*

Find the following antiderivatives or definite integrals.

**1.** (3pts) $\displaystyle\int \sqrt[6]{x^7}\,dx =$

**2.** (3pts) $\displaystyle\int e^{7x-1}\,dx =$

**3.** (6pts) $\displaystyle\int \frac{u^2-3u}{\sqrt{u}}\,du =$

**4.** (5pts) $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx =$

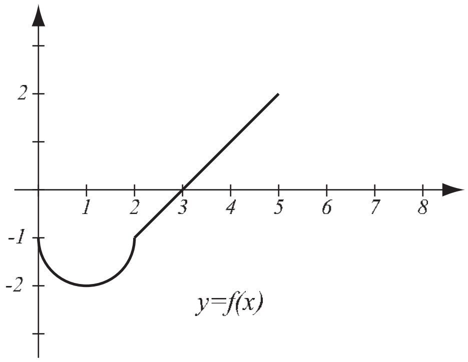
**5.** (6pts) $\displaystyle\int_0^{\frac{\pi}{3}} \sin\theta + \cos\theta\,d\theta =$

**6.** (6pts) Find $f(x)$ if $f'(x) = \dfrac{1}{\sqrt{x}} + \dfrac{1}{x}$ and $f(1) = 3$.

**7.** (15pts) The function $f(x) = \sqrt{x} - 1$ is given on the interval $[0, 3]$.
a) Write the Riemann sum $L_6$ for this function with six subintervals, taking sample points to be left endpoints. Do not evaluate the expression.
b) Illustrate with a diagram, where appropriate rectangles are clearly visible. What does $L_6$ represent?

**8.** (13pts) Find $\int_{-1}^{3} 4 - 2x \, dx$ in two ways (they'd better give you the same answer!):
a) Using the "area" interpretation of the integral. Draw a picture.
b) Using the Evaluation Theorem.

**9.** (10pts) The graph of a function $f$, consisting of lines and parts of circles, is shown. Evaluate the integrals.




$\int_0^2 f(x)\,dx =$

$\int_2^5 f(x)\,dx =$

$\int_0^5 f(x)\,dx =$

**10.** (16pts) Consider the integral $\displaystyle\int_{-1}^{2} -x^2 + 2x\,dx$.

a) Use the inequality $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$, where $m \leq f(x) \leq M$ on $[a,b]$, to give an estimate of the integral. (A graph of $-x^2+2x$ will help you find $m$ and $M$.)

b) Evaluate the integral and verify your estimate from a).

**11.** (7pts) Write using sigma notation:

$$\frac{9}{7} + \frac{16}{9} + \frac{25}{11} + \cdots + \frac{100}{21} =$$

**12.** (10pts) Helium is pumped into a balloon at rate $e^{-\frac{1}{8}t}$ cubic meters per minute.
a) Use the Net Change Theorem to find how much helium was added from $t = 0$ to $t = 4$ minutes.
b) If at time $t = 0$ there were 2 cubic meters helium in the balloon, how much is there at $t = 4$ minutes?

**Bonus.** (10pts) A car initially traveling at velocity 20 meters per second accelerates steadily for 5 seconds until it reaches velocity 30 meters per second. Find its position function to help you answer: how far did it travel while it was accelerating?

**Calculus 1 — Final Exam**
**MAT 250, Fall 2023 — D. Ivanšić**

**Name:** ____________________

*Show all your work!*

**1.** (16pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

$\lim_{x \to -3} f(x) =$

$\lim_{x \to 2-} f(x) =$

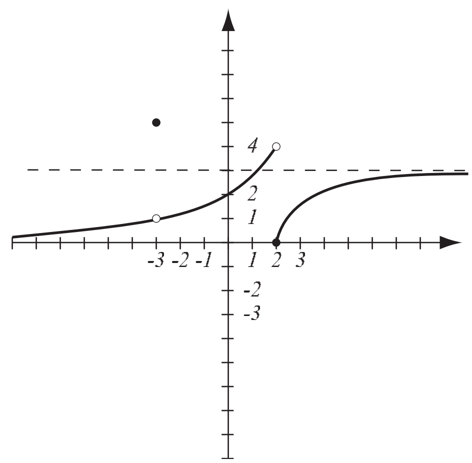
$\lim_{x \to 2+} f(x) =$

$\lim_{x \to 2} f(x) =$

$\lim_{x \to -\infty} f(x) =$

$\lim_{x \to \infty} f(x) =$

List points where $f$ is not continuous and explain why.




Find the following limits algebraically. Do not use L'Hospital's rule.

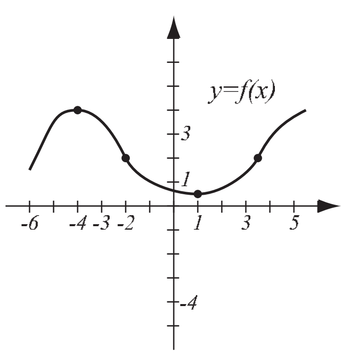
**2.** (6pts) $\lim_{x \to 2+} \dfrac{x+3}{10-5x} =$

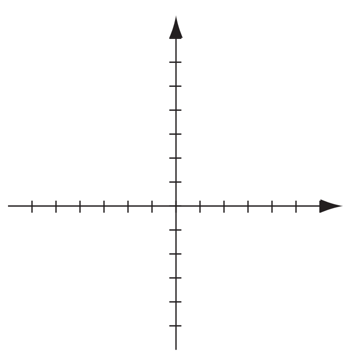
**3.** (6pts) $\lim_{x \to \infty} \dfrac{3x-11}{x^3-4x^2+2} =$

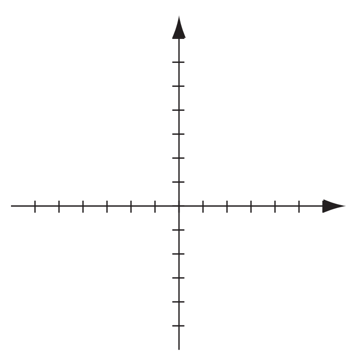
**4.** (8pts) Find $\lim_{x\to 0} x^2\left(1+\cos\frac{1}{x^3}\right)$. Use the theorem that rhymes with what water does when it is very cold.

**5.** (10pts) Consider the circle with equation $x^2+y^2=13$.
a) Use implicit differentiation to find the equation of the tangent line to the circle at point $(3,2)$.
b) Draw the circle and the tangent line.

**6.** (12pts) The graph of $f$ is given. Use it to draw the graphs of $f'$ and $f''$ in the coordinate systems provided. Pay attention to increasingness, decreasingness and concavity of $f$. The relevant special points have been highlighted.





**7.** (26pts) Let $f(x) = \ln|x^2 - 2x - 8|$. Draw an accurate graph of $f$ by following the guidelines.

a) State the domain of $f$ and its vertical asymptotes. Recall $\ln u$ is defined only for $u > 0$.

b) Find the intervals of increase and decrease, and local extremes. Note the absolute value goes away once you take the derivative.

c) Find the intervals of concavity and points of inflection.

d) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

e) Use information from a)–d) to sketch the graph.

**8.** (12pts) Let $f(\theta) = \sin\theta + \cos^2\theta$. Find the absolute minimum and maximum values of $f$ on the interval $[0, \pi]$.

**9.** (6pts) Find $f(x)$ if $f'(x) = 3\sqrt{x} + \dfrac{1}{1+x^2}$ and $f(1) = 4$.

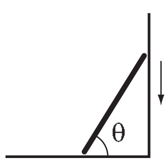
**10.** (10pts) Consider the integral $\displaystyle\int_0^1 e^x - 2\,dx$.

a) Use a picture and the "area" interpretation of the integral to determine whether this integral is positive or negative.

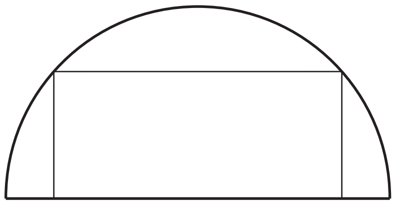
b) Use the Evaluation Theorem to find the integral and verify your conclusion from a).

**11.** (10pts) Helium is pumped into a balloon at rate $e^{-\frac{1}{2}t}$ cubic meters per minute.
a) Use the Net Change Theorem to find how much helium was added from $t = 0$ to $t = 6$ minutes.
b) If at time $t = 0$ there were 5 cubic meters helium in the balloon, how much is there at $t = 6$ minutes?

**12.** (12pts) A 3-meter ladder is sliding down the wall against which it is leaning. When the bottom of the ladder is 1 meter from the base of the wall, it is moving away from the wall at speed $\frac{1}{4}$ meters per second. How fast is the angle $\theta$ between the ladder and the floor changing at that moment?



**13.** (16pts) A half-circle of radius 1 meter is given. Among all rectangles that have one side on the diameter and the other two vertices on the half-circle, find the one with the greatest area.



**Bonus.** (10pts) Find the derivative and simplify until the bitter end. You will get the derivative of a simpler function. Which one?

$$\frac{d}{dx} \arctan \sqrt{\frac{1-x}{1+x}} =$$