Calculus 1 — Final Exam
MAT 250, Fall 2023 — D. Ivanšić

Name: Saul Ocean

Show all your work!

1. (16pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

$\lim_{x\to -3} f(x) =$ 1

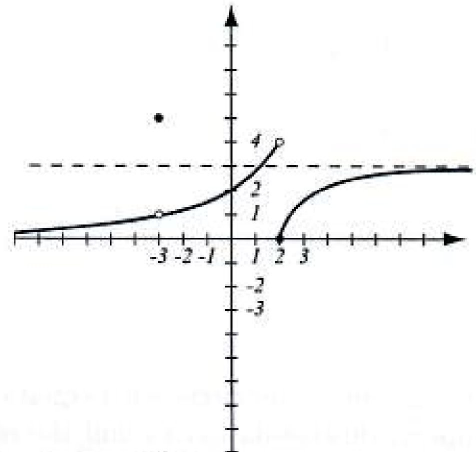
$\lim_{x\to 2-} f(x) =$ 4

$\lim_{x\to 2+} f(x) =$ 0

$\lim_{x\to 2} f(x) =$ DNE, one-sided limits not equal

$\lim_{x\to -\infty} f(x) =$ 0

$\lim_{x\to \infty} f(x) =$ 3



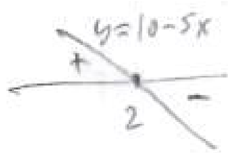
List points where f is not continuous and explain why.

Not cont. at $x=-3$, $\lim_{x\to -3} f(x) = 1 \neq 5 = f(-3)$

$x=2$, $\lim_{x\to 1} f(x)$ DNE

Find the following limits algebraically. Do not use L'Hospital's rule.

2. (6pts) $\lim_{x\to 2+} \frac{x+3}{10-5x} = \frac{5}{0-} = 5\cdot(-\infty) = -\infty$



3. (6pts) $\lim_{x\to\infty} \frac{3x-11}{x^3-4x^2+2} = \lim_{x\to\infty} \frac{x\left(3-\frac{11}{x}\right)}{x^3\left(1-\frac{4}{x}+\frac{2}{x^3}\right)} = \lim_{x\to\infty} \frac{1}{x^2} \frac{\left(3-\frac{11}{x}\right)}{\left(1-\frac{4}{x}+\frac{2}{x^3}\right)}$

$= 0\cdot\frac{3-0}{1-0+0} = 0\cdot 3 = 0$

4. (8pts) Find $\lim_{x\to 0} x^2\left(1+\cos\frac{1}{x^3}\right)$. Use the theorem that rhymes with what water does when it is very cold.

$-1 \le \cos\frac{1}{x^3} \le 1$

$0 \le 1+\cos\frac{1}{x^3} \le 2 \quad \cdot x^2$

$0 \le x^2\left(1+\cos\frac{1}{x^3}\right) \le 2x^2$

$\left.\begin{array}{l}\lim_{x\to 0} 0 = 0 \\ \lim_{x\to 0} 2x^2 = 0\end{array}\right\}$ Equal, so by squeeze theorem

$\lim_{x\to 0} x^2\left(1+\cos\frac{1}{x^2}\right) = 0$

5. (10pts) Consider the circle with equation $x^2+y^2=13$.
a) Use implicit differentiation to find the equation of the tangent line to the circle at point $(3,2)$.
b) Draw the circle and the tangent line.

a) $x^2+y^2=13 \quad \Big| \frac{d}{dx}$

$2x+2yy'=0$

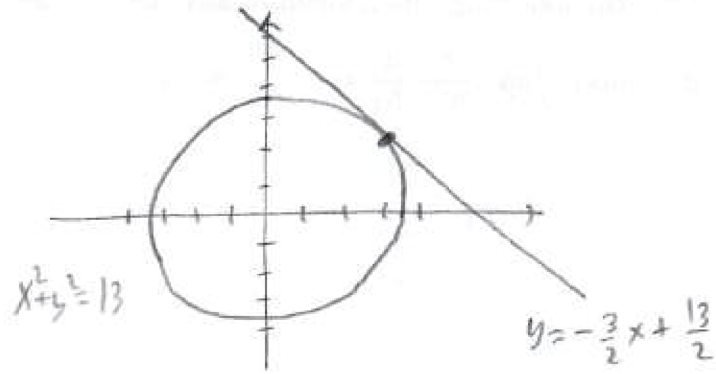
$2yy'=-2x$

$y'=-\frac{2x}{2y}=-\frac{x}{y}$

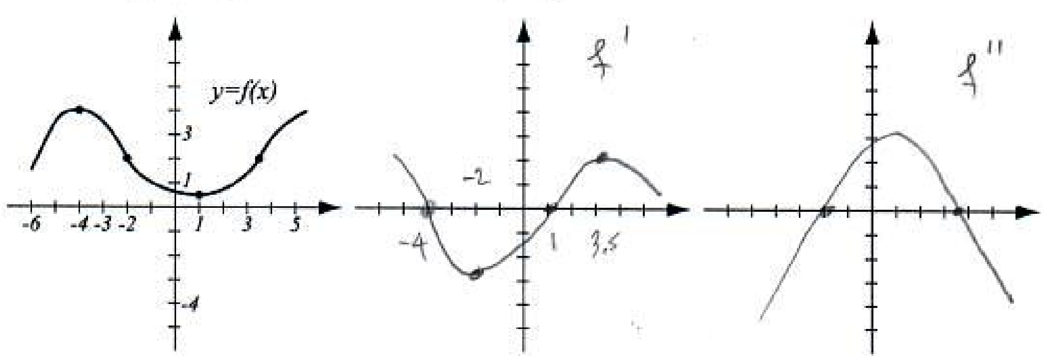
At $(3,2)$, $y'=-\frac{3}{2}$

$y-2=-\frac{3}{2}(x-3)$

$y=-\frac{3}{2}x+\frac{9}{2}+2=-\frac{3}{2}x+\frac{13}{2}$



6. (12pts) The graph of f is given. Use it to draw the graphs of f' and f'' in the coordinate systems provided. Pay attention to increasingness, decreasingness and concavity of f. The relevant special points have been highlighted.



7. (26pts) Let $f(x) = \ln|x^2 - 2x - 8|$. Draw an accurate graph of f by following the guidelines.

a) State the domain of f and its vertical asymptotes. Recall $\ln u$ is defined only for $u > 0$.

b) Find the intervals of increase and decrease, and local extremes. Note the absolute value goes away once you take the derivative.

c) Find the intervals of concavity and points of inflection.

d) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

e) Use information from a)–d) to sketch the graph.

a) Can't have $x^2-2x-8=0$

$(x-4)(x+2)=0$

$x=4,-2$

Vertical asymptotes $x=4,\ x=-2$

Because $\lim_{x\to -4} \ln|x^2-2x-8| = \ln 0^+ = -\infty$

$f'(x) = \frac{1}{x^2-2x-8}\cdot(2x-2) = 2\cdot\frac{x-1}{x^2-2x-8}$

$f''(x) = 2\cdot\frac{1\cdot(x^2-2x-8)-(x-1)\cdot 2(x-1)}{(x^2-2x-8)^2}$

$= 2\,\frac{x^2-2x-8-2(x^2-2x+1)}{(x^2-2x-8)^2}$

$= 2\,\frac{-x^2+2x-10}{(x^2-2x-8)^2}$

b) Crit. pts: $x-1=0$, $x=1$

$x^2-2x-8=0$, $x=4,-2$

$(x-4)(x+2)=0$

		-2		1		4	
$x-1$	−		−	0	+		+
x^2-2x-8	+	0	−		−	0	+
f'	−	ND	+	0	−	ND	+
f	↘		↗	loc max	↘		↗

c) 2nd order crit. pts:

$(x^2-2x-8)^2 \geq 0$ always, doesn't matter

$-x^2+2x-10=0$

$x^2-2x+10=0$

$x = \frac{-(-2)\pm\sqrt{(-2)^2-4\cdot1\cdot10}}{2\cdot1} = \frac{2\pm\sqrt{-36}}{2}$

No real sol.

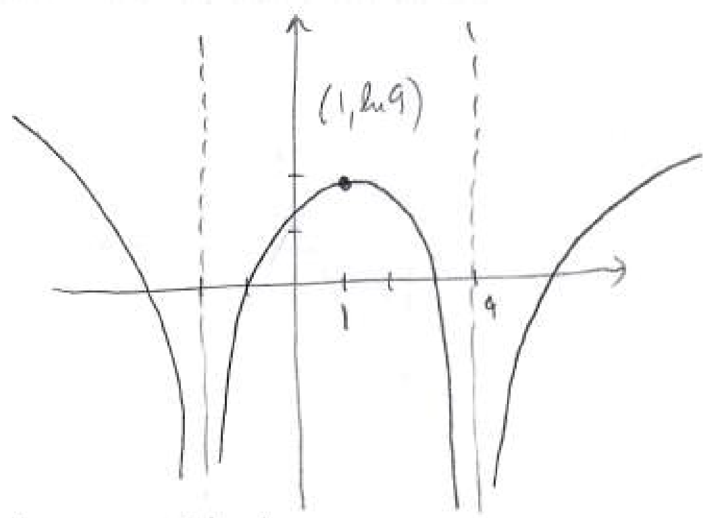
so $-x^2+2x-10<0$ for all x

$f''(x)<0$ always, so function is concave down always

d) $\lim_{x\to\infty} \ln|x^2-2x-8| = \ln \lim |x^2(1-\frac{2}{x}-\frac{8}{x^2})|$

$= \ln(\infty(1-0-0)) = \ln\infty = \infty$

Same for $x\to-\infty$



$\ln|1-2-8| = \ln|-9| \approx 2$

8. (12pts) Let $f(\theta) = \sin\theta + \cos^2\theta$. Find the absolute minimum and maximum values of f on the interval $[0, \pi]$.

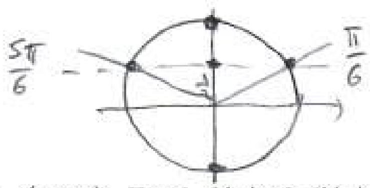
$f'(\theta) = \cos\theta + 2\cos\theta(-\sin\theta)$

$\cos\theta(1 - 2\sin\theta) = 0$

$\cos\theta = 0$ or $1 - 2\sin\theta = 0$.

$\theta = \frac{\pi}{2}$ $\quad$ $\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$



θ	$\sin\theta + \cos^2\theta$	
0	$0 + 1 = 1$	abs. min
π	$0 + 1 = 1$	
$\frac{\pi}{2}$	$1 + 0 = 1$	
$\frac{\pi}{6}$	$\frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$	abs max
$\frac{5\pi}{6}$	$\frac{1}{2} + \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$	

9. (6pts) Find $f(x)$ if $f'(x) = 3\sqrt{x} + \dfrac{1}{1+x^2}$ and $f(1) = 4$.

$f'(x) = 3x^{\frac{1}{2}} + \frac{1}{1+x^2}$

$f(x) = 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \arctan x + C$

$= 2x^{\frac{3}{2}} + \arctan x + C$

$4 = f(1) = 2 \cdot 1^{\frac{3}{2}} + \arctan 1 + C$

$4 = 2 + \frac{\pi}{4} + C, \quad C = 2 - \frac{\pi}{4}$

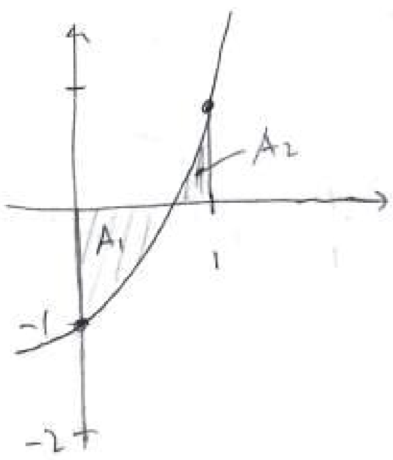
$f(x) = 2x^{\frac{3}{2}} + \arctan x + 2 - \frac{\pi}{4}$

10. (10pts) Consider the integral $\int_0^1 e^x - 2\,dx$.

a) Use a picture and the "area" interpretation of the integral to determine whether this integral is positive or negative.

b) Use the Evaluation Theorem to find the integral and verify your conclusion from a).

a)



$\int_0^1 e^x - 2\,dx = -A_1 + A_2 < 0$ since A_1 appears bigger

b) $\int_0^1 e^x - 2\,dx = (e^x - 2x)\Big|_0^1 = e^1 - e^0 - 2(1-0)$

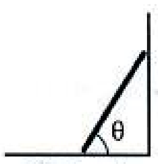
$= e - 1 - 2 = e - 3 < 0$

11. (10pts) Helium is pumped into a balloon at rate $e^{-\frac{1}{2}t}$ cubic meters per minute.
a) Use the Net Change Theorem to find how much helium was added from $t = 0$ to $t = 6$ minutes.
b) If at time $t = 0$ there were 5 cubic meters helium in the balloon, how much is there at $t = 6$ minutes?

a) $\int_0^6 e^{-\frac{1}{2}t}\,dt = \left.\frac{e^{-\frac{1}{2}t}}{-\frac{1}{2}}\right|_0^6 = -2\left(e^{-\frac{1}{2}\cdot 6} - e^{-\frac{1}{2}\cdot 0}\right) = -2\left(e^{-3} - 1\right) = 2\left(1 - \frac{1}{e^3}\right)$

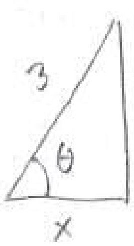
b) $V(6) = V(0) + V(6) - V(0) = 5 + 2\left(1 - \frac{1}{e^3}\right) = 7 - \frac{2}{e^3}$

12. (12pts) A 3-meter ladder is sliding down the wall against which it is leaning. When the bottom of the ladder is 1 meter from the base of the wall, it is moving away from the wall at speed $\frac{1}{4}$ meters per second. How fast is the angle θ between the ladder and the floor changing at that moment?



Know: $x' = \frac{1}{4}$ when $x = 1$

Need: θ' when $x = 1$



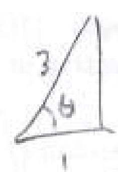
$\frac{x}{3} = \cos\theta$

$x = 3\cos\theta \quad \Big| \frac{d}{dt}$

$x' = -3\sin\theta\ \theta'$

$\theta' = -\frac{x'}{3\sin\theta}$

When $x = 1$



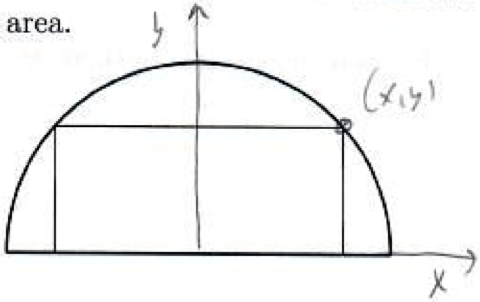
$\sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$

$\sin\theta = \frac{2\sqrt{2}}{3}$

$\theta' = -\frac{\frac{1}{4}}{3 \cdot \frac{2\sqrt{2}}{3}} = -\frac{1}{4} \cdot \frac{1}{2\sqrt{2}} = -\frac{1}{8\sqrt{2}}$ radians/sec

(θ is getting smaller, hence $\theta' < 0$)

13. (16pts) A half-circle of radius 1 meter is given. Among all rectangles that have one side on the diameter and the other two vertices on the half-circle, find the one with the greatest area.



$A = 2xy = 2x\sqrt{1-x^2}$

$x^2+y^2=1$, so $y = \pm\sqrt{1-x^2} = \sqrt{1-x^2}$ since $y>0$

$f(x) = A^2 = 4x^2(1-x^2)$

Job: Maximize $f(x)$ on $[0,1]$

$f'(x) = 4\left(2x(1-x^2)+x^2(-2x)\right) = 8x(1-x^2-x^2)$
$= 8x(1-2x^2)$

Crit. pts: $x=0$ or $1-2x^2=0$, $x^2=\frac{1}{2}$, $x=\pm\sqrt{\frac{1}{2}}=\pm\frac{\sqrt{2}}{2}$

x	$4x^2(1-x^2)$
0	0
1	0
$\sqrt{\frac{1}{2}}$	$4\cdot\frac{1}{2}(1-\frac{1}{2})=1$

Max area is for $x=\sqrt{\frac{1}{2}}$, $y=\sqrt{\frac{1}{2}}$

Bonus. (10pts) Find the derivative and simplify until the bitter end. You will get the derivative of a simpler function. Which one?

$$\frac{d}{dx}\arctan\sqrt{\frac{1-x}{1+x}} = \frac{1}{1+\left(\sqrt{\frac{1-x}{1+x}}\right)^2}\cdot\frac{1}{2\sqrt{\frac{1-x}{1+x}}}\cdot\frac{-1((1+x)-(1-x)\cdot 1)}{(1+x)^2} = \frac{1}{1+\frac{1-x}{1+x}}\cdot\frac{1}{\cancel{2}}\frac{\sqrt{1+x}}{\sqrt{1-x}}\cdot\frac{-\cancel{2}}{(1+x)^2}$$

$$= -\frac{1}{1+\frac{1-x}{1+x}}\cdot\frac{1}{1+x}\cdot\frac{\sqrt{1+x}}{1+x}\cdot\frac{1}{\sqrt{1-x}} = -\frac{1}{1+x+1-x}\cdot\frac{1}{\sqrt{1+x}}\cdot\frac{1}{\sqrt{1-x}} = -\frac{1}{2\sqrt{1-x^2}} = \left(\frac{1}{2}\arccos x\right)'$$