Calculus 1 — Exam 5
MAT 250, Fall 2023 — D. Ivanšić

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Show all your work!

Find the following antiderivatives or definite integrals.

1. (3pts) $\int \sqrt[6]{x^7}\,dx =$ $\dfrac{x^{\frac{13}{6}}}{\frac{13}{6}} = \dfrac{6}{13} x^{\frac{13}{6}} + C$

2. (3pts) $\int e^{7x-1}\,dx =$ $\frac{1}{7} e^{7x-1} + C$

3. (6pts) $\int \frac{u^2-3u}{\sqrt{u}}\,du =$ $\int \frac{u^2-3u}{u^{1/2}} = \int u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\,du = \dfrac{u^{\frac{5}{2}}}{\frac{5}{2}} - 3\dfrac{u^{\frac{3}{2}}}{\frac{3}{2}}$

$= \frac{2}{5} u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + C$

4. (5pts) $\int_0^1 \frac{1}{1+x^2}\,dx =$ $\arctan x \Big|_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

5. (6pts) $\int_0^{\frac{\pi}{3}} \sin\theta + \cos\theta\,d\theta =$ $(-\cos\theta + \sin\theta)\Big|_0^{\frac{\pi}{3}} = -\left(\cos\frac{\pi}{3} - \cos 0\right) + \left(\sin\frac{\pi}{3} - \sin 0\right)$

$= -\left(\frac{1}{2} - 1\right) + \left(\frac{\sqrt{3}}{2} - 0\right)$

$= \frac{\sqrt{3}+1}{2}$

6. (6pts) Find $f(x)$ if $f'(x) = \frac{1}{\sqrt{x}} + \frac{1}{x}$ and $f(1) = 3$.

$f'(x) = \frac{1}{\sqrt{x}} + \frac{1}{x} = x^{-\frac{1}{2}} + \frac{1}{x}$

$f(x) = \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} + \ln|x| = 2x^{\frac{1}{2}} + \ln|x| + C$

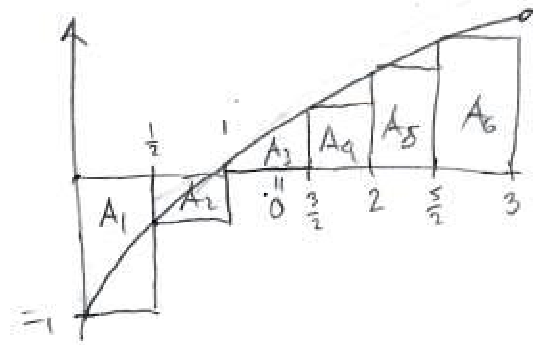
$3 = f(1) = 2 + \ln 1 + C =$

$3 = 2 + C, \quad C = 1$

$f(x) = 2\sqrt{x} + \ln|x| + 1$

7. (15pts) The function $f(x) = \sqrt{x} - 1$ is given on the interval $[0, 3]$.
a) Write the Riemann sum L_6 for this function with six subintervals, taking sample points to be left endpoints. Do not evaluate the expression.
b) Illustrate with a diagram, where appropriate rectangles are clearly visible. What does L_6 represent?

$\Delta x = \frac{3-0}{6} = \frac{1}{2}$



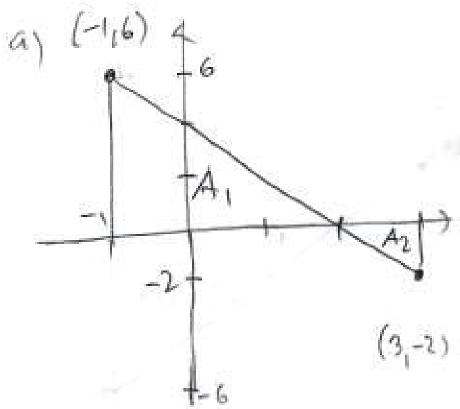
a) $L_6 = \frac{1}{2}\left(\sqrt{0}-1 + \sqrt{\frac{1}{2}}-1 + \sqrt{1}-1 + \sqrt{\frac{3}{2}}-1 + \sqrt{2}-1 + \sqrt{\frac{5}{2}}-1\right)$

$L_6 = -A_1 - A_2 + A_3 + A_4 + A_5 + A_6$

A_i = area of rectangle.

8. (13pts) Find $\int_{-1}^{3} -2x + 4\, dx$ in two ways (they'd better give you the same answer!):
a) Using the "area" interpretation of the integral. Draw a picture.
b) Using the Evaluation Theorem.

a)



areas of triangles

$\int_{-1}^{3} 2x-4\, dx = A_1 - A_2$

$= \frac{3 \cdot 6}{2} - \frac{1 \cdot 2}{2}$

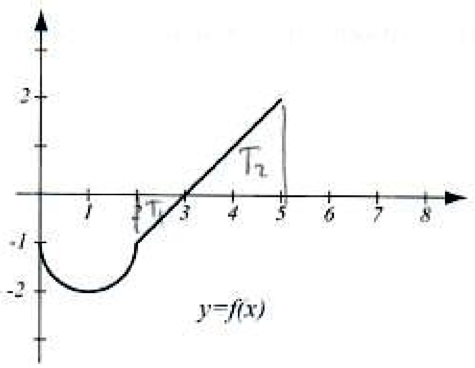
$= 9 - 1 = 8$

b) $\int_{-1}^{3} 4-2x\, dx = 4x - x^2 \Big|_{-1}^{3}$

$= 4(3-(-1)) - (9-1)$

$= 16 - 8 = 8$ same

9. (10pts) The graph of a function f, consisting of lines and parts of circles, is shown. Evaluate the integrals.



$\int_0^2 f(x)\,dx = -\left(\text{rectangle + half disk}\right)$

$-\left(2\cdot 1 + \frac{1}{2}\pi\cdot 1^2\right) = -2-\frac{\pi}{2}$

$\int_2^5 f(x)\,dx = -T_1 + T_2 = -\frac{1\cdot 1}{2} + \frac{2\cdot 2}{2} = \frac{3}{2}$

$\int_0^5 f(x)\,dx = -2-\frac{\pi}{2}+\frac{3}{2} = -\frac{1}{2}-\frac{\pi}{2}$

10. (16pts) Consider the integral $\displaystyle\int_{-1}^{2} -x^2+2x\,dx$.

a) Use the inequality $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$, where $m \le f(x) \le M$ on $[a, b]$, to give an estimate of the integral. (A graph of $-x^2+2x$ will help you find m and M.)

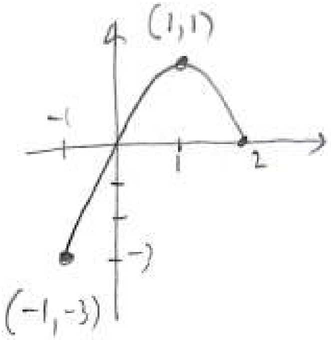
b) Evaluate the integral and verify your estimate from a).

a) $-x^2+2x=0$ $\quad f'(x)=0$

$x(-x+2)=0$ $\quad -2x+2=0$

$x=0,2$ $\quad x=1$

is peak



$M=1$

$m=-3$

$-3(2-(-1)) \le \int_{-1}^{2} -x^2+2x\,dx \le 1\cdot(2-(-1))$

$-9 \le \int_{-1}^{2} -x^2+2x\,dx \le 3$

b) $\int_{-1}^{2} -x^2+2x\,dx = -\frac{x^3}{3}+x^2\Big|_{-1}^{2}$

$= -\frac{1}{3}(8-(-1)) + 4-1 = -\frac{1}{3}\cdot 9+3 = 0$

Really: $-9 \le 0 \le 3$

11. (7pts) Write using sigma notation:

$\frac{9}{7}+\frac{16}{9}+\frac{25}{11}+\cdots+\frac{100}{21} = \sum_{i=3}^{10} \frac{i^2}{2i+1}$

12. (10pts) Helium is pumped into a balloon at rate $e^{-\frac{1}{8}t}$ cubic meters per minute.
a) Use the Net Change Theorem to find how much helium was added from $t = 0$ to $t = 4$ minutes.
b) If at time $t = 0$ there were 2 cubic meters helium in the balloon, how much is there at $t = 4$ minutes?

a) $V(4)-V(0) = \int_0^4 V'(t)\,dt = \int_0^4 e^{-\frac{1}{8}t}\,dt = \left.\frac{e^{-\frac{1}{8}t}}{-\frac{1}{8}}\right|_0^4 = \left.-8e^{-\frac{1}{8}t}\right|_0^4$

$= -8\left(e^{-\frac{1}{2}} - e^{0}\right) = -8\left(\frac{1}{\sqrt{e}} - 1\right) = 8 - \frac{8}{\sqrt{e}}$

b) $V(4) = V(0) + (V(4)-V(0)) = 2 + 8 - \frac{8}{\sqrt{e}} = 10 - \frac{8}{\sqrt{e}}$

Bonus. (10pts) A car initially traveling at velocity 20 meters per second accelerates steadily for 5 seconds until it reaches velocity 30 meters per second. Find its position function to help you answer: how far did it travel while it was accelerating?

acceleration $= \frac{30-20}{5} = 2\ m/s^2$

$a(t) = 2$

$v(t) = 2t + C \qquad v(0) = 20$

$20 = v(0) = 2\cdot 0 + C, \quad C = 20$

$v(t) = 2t + 20$

$s(t) = t^2 + 20t + D \qquad s(0) = 0$

$0 = s(0) = 0 + 0 + D, \quad D = 0$

$s(t) = t^2 + 20t$

$s(5) = 5^2 + 20\cdot 5 = 125$ meters

Car traveled 125 meters while accelerating

Alternative sol:

$s(5) - s(0) = \int_0^5 v(t)\,dt$

$= \int_0^5 20 + 2t\,dt$

$= \left.(t^2 + 20t)\right|_0^5 = 5^2 + 20\cdot 5 = 125$