Calculus 1 — Exam 4
MAT 250, Fall 2023 — D. Ivanšić

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Show all your work!

1. (32pts) Let $f(x) = \ln(x^2 - 2x + 5)$. The domain of this function is all real numbers (you do not have to verify this). Draw an accurate graph of f by following the guidelines.

a) Find the intervals of increase and decrease, and local extremes.

b) Find the intervals of concavity and points of inflection.

c) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

d) Use information from a)–c) to sketch the graph.

$$f'(x) = \frac{1}{x^2-2x+5}\cdot(2x-2) = 2\cdot\frac{x-1}{x^2-2x+5}$$

$$f''(x) = 2\,\frac{1\cdot(x^2-2x+5)-(x-1)(2x-2)}{(x^2-2x+5)^2}$$

$$= 2\cdot\frac{x^2-2x+5-(2x^2-4x+2)}{(x^2+2x+5)^2}$$

$$= 2\cdot\frac{-x^2+2x+3}{(x^2-2x+5)^2}$$

a) $f'(x) = \dfrac{2(x-1)}{x^2-2x+5}$

Crit pts: $x-1=0 \quad x=1$

$x^2-2x+5=0$

$$x = \frac{-(-2)\pm\sqrt{(-2)^2-4\cdot1\cdot5}}{2} = \frac{2\pm\sqrt{-16}}{2}$$

no real solutions, so graph is

thus $x^2-2x+5 \geq 0$ always

		1	
f'	$-$	0	$+$
f	$\searrow$	loc. min	$\nearrow$

$y = x-1$

b) $f''(x) = \dfrac{2(-x^2+2x+3)}{(x^2-2x+5)^2} \leftarrow \geq 0$

Crit pts: $-x^2+2x+3=0$

$x^2-2x-3=0$

$(x-3)(x+1)=0$

$x=-1, 3$

No crit pts from denominator (from before)

$y=-x^2+2x+3$

		-1		3	
f''	$-$	0	$+$	0	$-$
f	CD	IP	CU	IP	CD

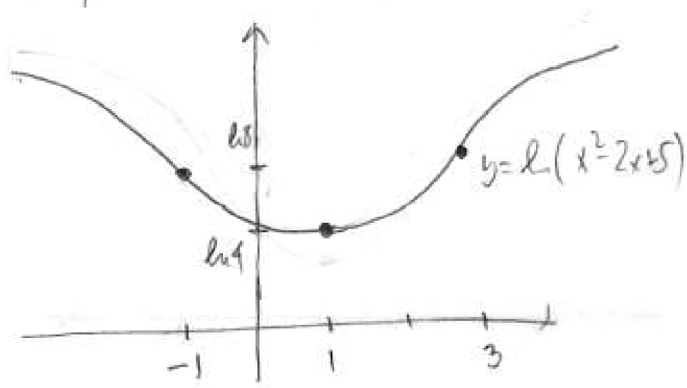
c) $\lim\limits_{x\to\infty} \ln(x^2-2x+5) = \lim\limits_{x\to\infty} \ln\left(x^2\left(1-\frac{2}{x}+\frac{5}{x^2}\right)\right)$

$= \ln(\infty\cdot 1) = \infty$

$\lim\limits_{x\to-\infty} f(x) = \infty$ in same way

d)

x	$\ln(x^2-2x+5)$
-1	$\ln(1+2+5) = \ln 8 = \ln 4^{\frac{3}{2}} = \frac{3}{2}\ln 4$
1	$\ln(1-2+5) = \ln 4$
3	$\ln(9-6+5) = \ln 8 =$



2. (18pts) Let $f(x) = \sqrt{2}\sin^2\theta + \frac{4}{3}\cos^3\theta$. Find the absolute minimum and maximum values of f on the interval $[0, \pi]$.

$$f'(x) = \sqrt{2}\cdot 2\sin\theta\cos\theta + \cancel{\tfrac{4}{3}}\cdot\cancel{3}\cos^2\theta\cdot(-\sin\theta)$$

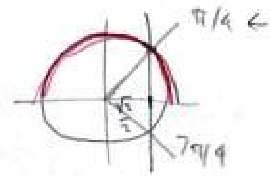
$$= 2\sin\theta\cos\theta\left(\sqrt{2} - 2\cos\theta\right) = 0$$

$\sqrt{2} - 2\cos\theta = 0$

$\sqrt{2} = 2\cos\theta$

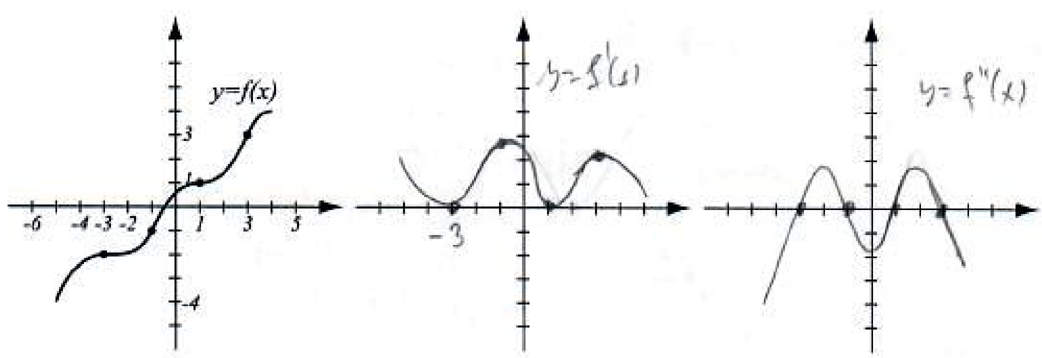
$\sin\theta = 0 \qquad \cos\theta = 0 \qquad \cos\theta = \frac{\sqrt{2}}{2}$

$\theta = 0, \pi \qquad \theta = \frac{\pi}{2} \qquad \theta = \frac{\pi}{4}$



θ	$\sqrt{2}\sin^2\theta + \frac{4}{3}\cos^3\theta$
0	$\sqrt{2}\cdot 0 + \frac{4}{3} = \frac{4}{3}$
π	$\sqrt{2}\cdot 0 - \frac{4}{3} = -\frac{4}{3}$ abs min $\left(\sqrt{2} = 1.41 > \frac{4}{3}\right)$
$\frac{\pi}{2}$	$\sqrt{2} + 0 = \sqrt{2}$ abs max
$\frac{\pi}{4}$	$\sqrt{2}\cdot\left(\frac{\sqrt{2}}{2}\right)^2 + \frac{4}{3}\left(\frac{\sqrt{2}}{2}\right)^3 = \frac{2\sqrt{2}}{4} + \frac{4}{3}\cdot\frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{3} = \frac{5\sqrt{2}}{6} < \sqrt{2}$

3. (14pts) The graph of f is given. Use it to draw the graphs of f' and f'' in the coordinate systems provided. Pay attention to increasingness, decreasingness and concavity of f. The relevant special points have been highlighted.



4. (14pts) Consider $f(x) = \dfrac{1}{x+1}$ on the interval $[0, 2]$.
a) Verify that the function satisfies the assumptions of the Mean Value Theorem.
b) Find all numbers c that satisfy the conclusion of the Mean Value Theorem.

a) Since $1 \le x+1 \le 3$, $x+1 \neq 0$ so $\frac{1}{x+1}$ is continuous on $[0,2]$
differentiable on $(0,2)$

b) $\dfrac{f(2)-f(0)}{2-0} = \dfrac{\frac{1}{3} - \frac{1}{1}}{2-0} = \dfrac{-\frac{2}{3}}{2} = -\frac{1}{3}$

$f'(x) = -\dfrac{1}{(x+1)^2}$

$-\dfrac{1}{(x+1)^2} = -\dfrac{1}{3}$

$(x+1)^2 = 3$

$x+1 = \pm\sqrt{3}$

$x = -1 \pm \sqrt{3}$

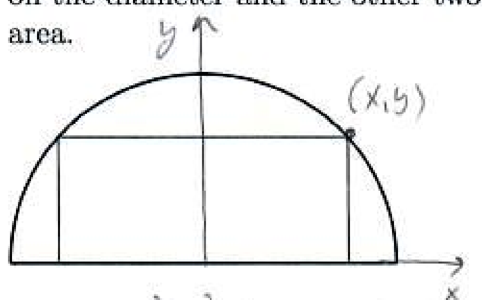
$-1-\sqrt{3} < 0$ so not in $(0,2)$

$-1+\sqrt{3} \approx -1 + 1.7 = 0.7$

which is in $(0,2)$

$\left(\begin{array}{l} 1 < \sqrt{3} < 2 \quad |-1 \\ 0 < \sqrt{3}-1 < 1 \end{array} \right)$

5. (22pts) A half-circle of radius 1 meter is given. Among all rectangles that have one side on the diameter and the other two vertices on the half-circle, find the one with the greatest area.



$x^2 + y^2 = 1$

$y^2 = 1 - x^2$

$y = \pm\sqrt{1-x^2} = \sqrt{1-x^2}$

since it is upper half-circle

$A = 2xy = 2x\sqrt{1-x^2}$

Consider $f(x) = A^2 = 4x^2(1-x^2)$

Job: maximize $f(x)$ on $[0,1]$.

$f'(x) = 4(2x(1-x^2) + x^2(-2x)) = 8x(1-x^2-x^2)$

$= 8x(1-2x^2)$

$8x(1-2x^2) = 0$

$x = 0$ or $1 - 2x^2 = 0$

$x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$

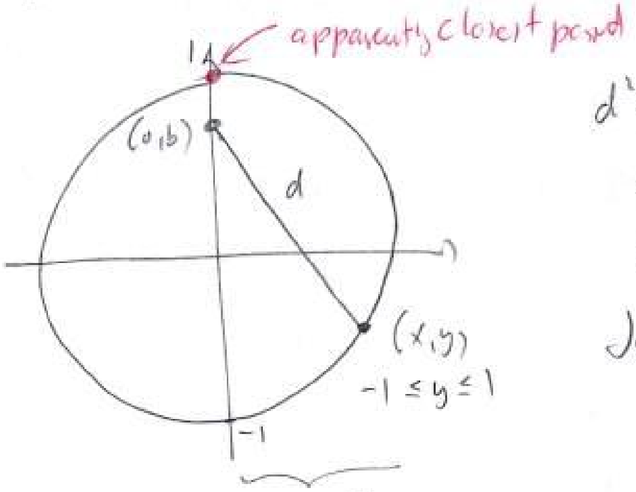
$(-\frac{1}{\sqrt{2}}$ not in interval$)$

x	$4x^2(1-x^2)$
0	0
1	0
$\frac{1}{\sqrt{2}}$	$4 \cdot \frac{1}{2}(1-\frac{1}{2}) = 1$ abs max

Bonus. (10pts) Let $0 < b < 1$.

a) Among all points on the unit circle, mark the point that you think is closest to point $(0, b)$.

b) Using calculus, show that this point is indeed the closest one to point $(0, b)$.



$$d^2 = (x-0)^2 + (y-b)^2$$
$$= x^2 + y^2 - 2yb + b^2$$
$$= 1 - 2yb + b^2 = f(y)$$

Job: Minimize $f(y)$ on $[-1,1]$

$f'(y) = -2b$ $\quad -2b = 0$ no sol.

due to symmetry, we can consider just the right half of circle

y	$f(y) = 1 - 2yb + b^2$
-1	$1 + 2b + b^2$
1	$1 - 2b + b^2$ abs min

Closest point really is $(0,1)$.