Calculus 1 — Exam 3
MAT 250, Fall 2023 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (4pts) $\dfrac{d}{dx}3^{\tan x} =$ $3^{\tan x}\cdot \ln 3 \cdot \sec^2 x$

2. (6pts) $\dfrac{d}{du}(u^2-2u+2)e^u =$ $(2u-2)e^u + (u^2-2u+2)e^u$

$= (2u-2+u^2-2u+2)e^u = u^2 e^u$

3. (7pts) $\dfrac{d}{du}\dfrac{e^u+u}{e^u-u} =$ $\dfrac{(e^u+1)(e^u-u)-(e^u+u)(e^u-1)}{(e^u-u)^2} = \dfrac{\cancel{e^{2u}}+e^u-ue^u-\cancel{u}-(\cancel{e^{2u}}+ue^u-e^u-\cancel{u})}{(e^u-u)^2}$

$= \dfrac{2e^u-2ue^u}{(e^u-u)^2} = \dfrac{2e^u(1-u)}{(e^u-u)^2}$

4. (7pts) $\dfrac{d}{dx}\ln\left(\sin^2 x\cos^2 x\right) =$ $\dfrac{d}{dx}\left(\ln \sin^2 x + \ln \cos^2 x\right) = \dfrac{d}{dx}\left(2\ln\sin x + 2\ln(\cos x)\right)$

$= 2\cdot\dfrac{1}{\sin x}\cdot\cos x + 2\,\dfrac{1}{\cos x}\cdot(-\sin x) = 2(\cot x - \tan x)$

5. (7pts) $\dfrac{d}{dt}\left(t\arccos t - \sqrt{1-t^2}\right) =$ $1\cdot\arccos t + t\cdot\left(-\dfrac{1}{\sqrt{1-t^2}}\right) - \dfrac{1}{\cancel{2}\sqrt{1-t^2}}\cdot(-\cancel{2}t)$

$= \arccos t - \dfrac{t}{\sqrt{1-t^2}} + \dfrac{t}{\sqrt{1-t^2}} = \arccos t$

6. (9pts) Use logarithmic differentiation to find the derivative of $y = x^{\arctan x}$.

$y = x^{\arctan x}$

$\dfrac{d}{dx}\Big|\ \ln y = \arctan x \ln x$

$\dfrac{y'}{y} = \dfrac{1}{1+x^2}\ln x + \arctan x\cdot\dfrac{1}{x}$

$y' = y(\ \text{—}\ )$

$= x^{\arctan x}\left(\dfrac{\ln x}{1+x^2} + \dfrac{\arctan x}{x}\right)$

Find the limits algebraically. Graphs of basic functions will help, as will L'Hospital's rule, where appropriate.

7. (2pts) $\lim_{x\to 0+} \ln(2x) =$ $\ln(0+) = -\infty$

8. (6pts) $\lim_{x\to\infty} e^{-\frac{x^2+1}{x+3}} =$ $e^{-\lim_{x\to\infty}\frac{x^2+1}{x+3}} = e^{-\lim_{x\to\infty}\frac{x^2(1+\frac{1}{x^2})}{x(1+\frac{3}{x})}} = e^{-\lim_{x\to\infty} x\cdot\frac{1+\frac{1}{x^2}}{1+\frac{3}{x}}}$

$= e^{-\infty\cdot\frac{1+0}{1+0}} = e^{-\infty} = 0$

9. (7pts) $\lim_{x\to 0} \dfrac{\overset{1-1=0}{\cos(4x) - 1}}{\underset{0}{x^2}} \overset{L'H}{=} \lim_{x\to 0} \dfrac{\overset{\to 0}{-\sin(4x)\cdot 4}}{\underset{\to 0}{2x}} \overset{L'H}{=} \lim_{x\to 0} \dfrac{-\cos(4x)\cdot 16}{2}$

$= -8\cos 0 = -8$

10. (9pts) $\lim_{x\to 0+} \overset{0\cdot\infty}{x(\ln x)^2} =$ $\lim_{x\to 0+} \dfrac{\overset{\to\infty}{(\ln x)^2}}{\underset{\to\frac{1}{0+}=\infty}{\frac{1}{x}}} = \lim_{x\to 0+} \dfrac{2\ln x\cdot\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+} \dfrac{\overset{\to -\infty}{2\ln x}}{\underset{\to -\infty}{-\frac{1}{x}}} \overset{L'H}{=} \lim_{x\to 0+} \dfrac{\frac{2}{x}}{\frac{1}{x^2}}$

$= \lim_{x\to 0+} \frac{2}{x}\cdot\frac{x^2}{1} = \lim_{x\to 0+} 2x = 0$

11. (8pts) $\lim_{x\to 0+} \overset{1^\infty}{(1 - 3x)^{\frac{1}{x}}} =$ e^{-3}

$y = (1-3x)^{\frac{1}{x}}$

$\ln y = \frac{1}{x}\ln(1-3x)$

$\lim_{x\to 0+} \dfrac{\overset{\to \ln 1 = 0}{\ln(1-3x)}}{\underset{\to 0}{x}} = \lim_{x\to 0+} \dfrac{\frac{1}{1-3x}(-3)}{1} = -\frac{3}{1-0} = -3$

12. (11pts) Let $f(x) = \sqrt{x}$.
a) Write the linearization of $f(x)$ at $a = 4$.
b) Use the linearization to estimate $\sqrt{4.5}$.
c) In the same coordinate system, draw graphs of the function and the linearization and determine if the estimate in b) is an overestimate or underestimate of $\sqrt{4.5}$.

a) $f'(x) = \frac{1}{2\sqrt{x}}$

$f(4) = 2$

$f'(4) = \frac{1}{4}$

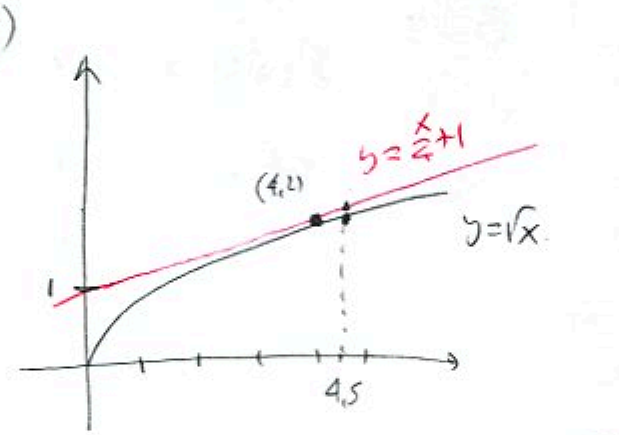
$L(x) = 2 + \frac{1}{4}(x-4) = 1 + \frac{x}{4}$

b) $L(4.5) = 2 + \frac{1}{4}(4.5-4)$

$= 2 + \frac{1}{4} \cdot \frac{1}{2}$

$= 2\frac{1}{8} = \frac{17}{8}$

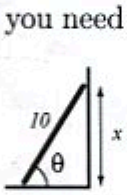
c)



$L(4.5)$ is an overestimate of $\sqrt{4.5}$

(above $x = 4.5$, point on linearization is above point on $f(x)$)

13. (10pts) A 10-foot ladder leans against the wall. In an effort to compute the angle θ that the ladder subtends with the floor, we measure the distance x from the floor to the top of the ladder and find it to be 8 feet with maximum error in measurement $\frac{1}{2}$ inch. Use differentials to estimate the maximum possible error when computing the angle θ. (Since you need to express angle θ as a function of x, an inverse trigonometric function is involved.)



$\sin\theta = \frac{x}{10}$

$\theta = \arcsin\frac{x}{10}$

$d\theta = \frac{1}{\sqrt{1-(\frac{x}{10})^2}} \cdot \frac{1}{10}\,dx$

When $x = 8$, $dx = \frac{1}{2} \cdot \frac{1}{12}$, $d\theta = \frac{1}{\sqrt{1-(\underset{\uparrow \frac{4}{5}}{\frac{8}{10}})^2}} \cdot \frac{1}{10} \cdot \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{\sqrt{1-\frac{16}{25}}} \cdot \frac{1}{240} = \frac{1}{\sqrt{\frac{9}{25}}} \cdot \frac{1}{240}$

$= \frac{\cancel{5}^{1}}{3} \cdot \frac{1}{\cancel{240}_{48}} = \frac{1}{144}$ rad

14. (7pts) Let $f(x) = 2^x + x$. Use the theorem on derivatives of inverses to find $(f^{-1})'(11)$.

$f'(x) = 2^x \cdot \ln 2 + 1$

$(f^{-1})'(11) = \dfrac{1}{f'(f^{-1}(11))} = \dfrac{1}{2^3 \cdot \ln 2 + 1} = \dfrac{1}{8\ln 2 + 1}$

$f^{-1}(11)$ is sol to $f(x) = 11$

$2^x + x = 11$

guess: $x = 3$

Bonus. (10pts) Find the derivative and simplify until the bitter end. You will get the derivative of a simpler function. Which one?

$$\frac{d}{dx} \arctan \sqrt{\frac{1-x}{1+x}} = \frac{1}{1 + \sqrt{\frac{1-x}{1+x}}^{\,2}} \cdot \frac{1}{2\sqrt{\frac{1-x}{1+x}}} \cdot \underbrace{\frac{(-1)(1+x) - (1-x)\cdot 1}{(1+x)^2}}_{\left(\frac{1-x}{1+x}\right)'}$$

$$= \frac{1}{1 + \frac{1-x}{1+x}} \cdot \frac{\sqrt{1+x}}{\cancel{2}\sqrt{1-x}} \cdot \frac{-\cancel{2}}{(1+x)^2} = \frac{1}{\left(1 + \frac{1-x}{1+x}\right)(1+x)} \cdot \frac{\cancel{\sqrt{1+x}}}{\underset{\sqrt{1+x}}{\cancel{1+x}}} \cdot \frac{-1}{\sqrt{1-x}}$$

$$= \frac{1}{1+x+1-x} \cdot \frac{1}{\sqrt{1+x}} \cdot \frac{-1}{\sqrt{1-x}} = -\frac{1}{2\sqrt{1-x^2}} = \left(\frac{1}{2} \arccos x\right)'$$