Calculus 1 — Exam 2
MAT 250, Fall 2023 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\frac{d}{dx}\left(4x^7 - \frac{5}{x^6} + \sqrt[3]{x^5} + c^{\frac{3}{2}}\right) = \frac{d}{dx}\left(4x^7 - 5x^{-6} + x^{\frac{5}{3}} + c^{\frac{3}{2}}\right)$ ← constant, der = 0

$= 28x^6 + 30x^{-7} + \frac{5}{3}x^{\frac{2}{3}}$

2. (6pts) $\frac{d}{dx}\frac{\sqrt{x}}{x^2+1} = \frac{\frac{1}{2\sqrt{x}}(x^2+1) - \sqrt{x}\cdot 2x}{(x^2+1)^2} \cdot \frac{2\sqrt{x}}{2\sqrt{x}} = \frac{x^2+1-4x^2}{2\sqrt{x}(x^2+1)^2} = \frac{1-3x^2}{2\sqrt{x}(x^2+1)^2}$

3. (6pts) $\frac{d}{du}(u+2)^4(u-1)^3 = 4(u+2)^3(u-1)^3 + (u+2)^4\cdot 3(u-1)^2$

$= (u+2)^3(u-1)^2\left(4(u-1) + 3(u+2)\right)$

$= (u+2)^3(u-1)^2(7u+2)$

4. (5pts) $\frac{d}{d\theta}\frac{1-\sin\theta}{1+\sin\theta} = \frac{-\cos\theta(1+\sin\theta) - (1-\sin\theta)\cos\theta}{(1+\sin\theta)^2}$

$= \frac{-\cos\theta - \cancel{\sin\theta\cos\theta} - \cos\theta + \cancel{\sin\theta\cos\theta}}{(1+\sin\theta)^2} = -\frac{2\cos\theta}{(1+\sin\theta)^2}$

5. (6pts) $\frac{d}{dx}\sqrt{x+\sqrt{\cancel{\cos} \tan x}} = \frac{1}{2\sqrt{x+\sqrt{\tan x}}}\cdot\left(1 + \frac{1}{2\sqrt{\tan x}}\cdot\sec^2 x\right)$

$= \frac{1 + \frac{\sec^2 x}{2\sqrt{\tan x}}}{2\sqrt{x+\sqrt{\tan x}}}$

6. (7pts) The limit at right represents a derivative $f'(a)$.
a) State f and a.
b) To find the limit, evaluate $f'(a)$ using differentiation rules.

$$\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{4}+h\right) - \frac{\sqrt{2}}{2}}{h}$$

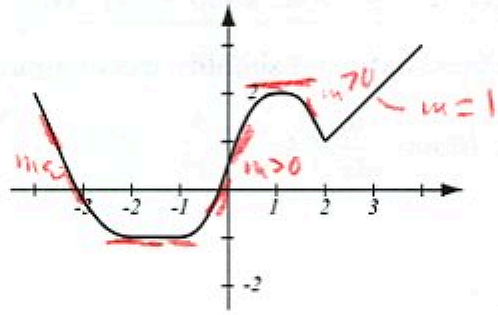
$= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

a) $a = \frac{\pi}{4}$, $f(x) = \cos x$

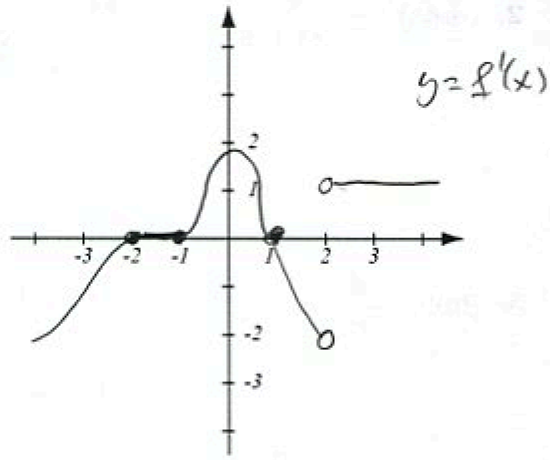
b) $f'(x) = -\sin x$, $f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

7. (10pts) The graph of the function $f(x)$ is shown at right.

a) Where is $f(x)$ not differentiable? Why?
b) Use the graph of $f(x)$ to draw an accurate graph of $f'(x)$.



a) At $x=2$, because it is a sharp point.



8. (12pts) Let $f(x) = \dfrac{2}{x}$.
a) Use the limit definition of the derivative to find the derivative of the function.
b) Check your answer by taking the derivative of f using differentiation rules.
c) Write the equation of the tangent line to the curve $y = f(x)$ at point $(2, 1)$.

a) $$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\frac{2}{x}-\frac{2}{a}}{x-a} = \lim_{x\to a} \frac{\frac{2a-2x}{xa}}{x-a} = \lim_{x\to a} \frac{2(\overset{(-1)}{\cancel{a-x}})}{xa}\cdot\frac{1}{\cancel{x-a}}$$

$$= \lim_{x\to a} -\frac{2}{xa} = -\frac{2}{a^2}, \quad \text{so } f'(x) = -\frac{2}{x^2}$$

b) $f(x) = 2x^{-1}$, so $f'(x) = 2(-1)x^{-2} = -\dfrac{2}{x^2}$, same

c) $f'(2) = -\dfrac{2}{2^2} = -\dfrac{1}{2}$ $\qquad y-1 = -\frac{1}{2}(x-2)$

$y = -\frac{1}{2}x + 1 + 1$

$y = -\frac{1}{2}x + 2$

9. (10pts) Let $g(x) = f(x)^2$ and $h(x) = \dfrac{f(x^2)}{x}$.

a) Find the general expressions for $g'(x)$ and $h'(x)$.

b) Use the table of values at right to find $g'(1)$ and $h'(2)$.

x	1	2	3	4
$f(x)$	2	-3	-1	3
$f'(x)$	-2	5	-4	1

a) $g'(x) = 2f(x)\cdot f'(x)$

$h'(x) = \dfrac{f'(x^2)\cdot 2x\cdot x - f(x^2)\cdot 1}{x^2} = \dfrac{2x^2f'(x^2) - f(x^2)}{x^2}$

b) $g'(1) = 2f(1)\,f'(1) = 2\cdot 2\cdot(-2) = -8$

$h'(2) = \dfrac{2\cdot 2^2\cdot f'(4) - f(4)}{2^2} = \dfrac{8\cdot 1 - 3}{4} = \dfrac{5}{4}$

10. (7pts) An arrow shot upwards has position given by the formula $s(t) = -5t^2 + 50t$.

a) Write the formula for the velocity of the arrow at time t.

b) What is the velocity of the arrow when it is at height 120 meters on the way up? On the way down?

a) $v(t) = -10t + 50$

b) $s(t) = 120$

$-5t^2 + 50t = 120$

$5t^2 - 50t + 120 = 0 \quad | \div 5$

$t^2 - 10t + 24 = 0$

$(t-6)(t-4) = 0$

$t = 4, 6$

$v(4) = -10\cdot 4 + 50 = 10$ m/s a way up

$v(6) = -10\cdot 6 + 50 = -10$ m/s on way down

11. (11pts) Use implicit differentiation to find y' in general, and then at point $\left(0, \dfrac{1}{2}\right)$ in particular.

$x^2 + y^2 = 4x^4 + 4y^4 + 8x^2y^2 \quad | \frac{d}{dx}$

$2x + 2yy' = 16x^3 + 16y^3\cdot y' + 8(2xy^2 + x^2\cdot 2yy')$

$2x + 2yy' = 16x^3 + 16y^3y' + 16xy^2 + 16x^2yy' \quad | \div 2$

$x - 8x^3 - 8xy^2 = 8y^3y' + 8x^2yy' - yy'$

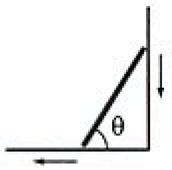
$x - 8x^3 - 8xy^2 = y'(8y^3 + 8x^2y - y)$

$y' = \dfrac{x - 8x^3 - 8xy^2}{8y^3 + 8x^2y - y}$

When $x = 0, y = \frac{1}{2}$ get

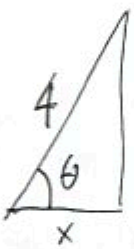
$y' = \dfrac{0}{1 + 0 - \frac{1}{2}} = 0$

12. (14pts) A 4-meter ladder is sliding down the wall against which it is leaning. When the bottom of the ladder is 1 meter from the base of the wall, it is moving away from the wall at speed $\frac{1}{5}$ meters per second. How fast is the angle θ between the ladder and the floor changing at that moment?



Know: $\frac{dx}{dt} = \frac{1}{5}$

Need $\frac{d\theta}{dt}$ when $x = 1$

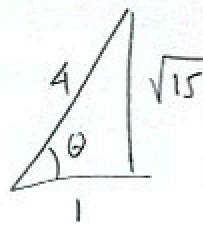


$\frac{x}{4} = \cos\theta$

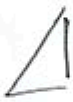
$x = 4\cos\theta$

$x' = -4\sin\theta \cdot \theta'$

$\theta' = -\frac{x'}{4\sin\theta}$

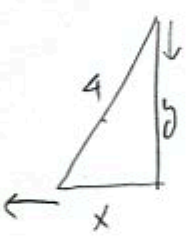


$\sin\theta = \frac{\sqrt{15}}{4}$



when $x=1$ $\quad \theta' = -\frac{\frac{1}{5}}{4\cdot\sin\theta} = -\frac{1}{20\cdot\frac{\sqrt{15}}{4}} = -\frac{1}{5\sqrt{15}}$ rad/sec.

Bonus. (10pts) Consider the sliding ladder from above. Show that, as the ladder slides down, the top of the ladder has to leave the wall before it hits the ground by doing this:
1) Suppose the top of the ladder stays on the wall. If x and y are distances from the bottom and top of the ladder to the base of the wall, respectively, find the general formula for y' in terms of x and x' (angle does not play a part here).
2) Assuming bottom of ladder slides at a constant $\frac{1}{5}$ meters per second away from the wall, What happens to y' as $x \to 4$? Is this possible in reality?



1) $x^2 + y^2 = 16 \quad \Big| \frac{d}{dt}$

$2xx' + 2yy' = 0$

$2yy' = -2xx'$

$y' = -\frac{2xx'}{2\cdot y} = -\frac{xx'}{\sqrt{16-x^2}}$

2) $\lim_{x\to 4^-} -\frac{xx'}{\sqrt{16-x^2}} = -\frac{4\cdot\frac{1}{5}}{\sqrt{0}} = -\frac{\frac{4}{5}}{0^+} = -\infty$

Velocity of any particle cannot be ∞, so assumption that ladder stays on wall is incorrect.