5.0 (Appendix B) Sigma Notation

Sigma notation is a shorthand way to write sums.

Definition. Let $a_m, a_{m+1}, \dots, a_{n-1}, a_n$ be real numbers, $m \le n$. We define

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n.$$

Often the term a_i has form $f(i)$ for some function f.

Examples. Write out the following sums and compute the sums where it is simple to do.

$$\sum_{i=3}^{7} \frac{1}{i^2} =$$

$$\sum_{i=0}^{5} \frac{1}{2^i} =$$

$$\sum_{i=2}^{6} 2 =$$

$$\sum_{i=0}^{5} \sin\left(\frac{i\pi}{2}\right) =$$

$$\sum_{i=3}^{6} (-1)^i x^i =$$

Theorem.

$$\sum_{i=m}^{n} c a_i = c \sum_{i=m}^{n} a_i \qquad \sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

Proof.

Examples.

$$\sum_{i=1}^{n} 1 =$$

$$\sum_{i=1}^{n} i =$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

5.1 Areas and Distances

The problem in this chapter is to find the area under the graph of a function $y = f(x)$, above the x-axis, and between vertical lines $x = a$ and $x = b$.

Example. Find the area under the parabola $y = x^2$ between lines $x = 0$ and $x = 2$.
We can't do this exactly, so we approximate the area with figures whose area we know: rectangles and trapezoids.

Note. The terms for rectangles — left, right, midpoint — refers to where f is evaluated to get height of rectangle: at the left, right endpoint, or at the midpoint of the subinterval.

We see $L_6 \leq A \leq T_6 \leq R_6$. Note that $T_6 = \dfrac{L_6 + R_6}{2}$. (It's harder to see where M_6 fits in.)

Investigate values for L_n, R_n, T_n and M_n as n increases.

n	L_n	R_n	T_n	M_n
10				
50				
100				
500				

It appears that L_n, R_n, T_n, M_n all approach the same number, the area of the region.

We show that $\lim_{n\to\infty} R_n$ exists.

More generally, to find area under a curve:

1) Subdivide the interval $[a, b]$ into equal-length subintervals of length $\Delta x = \dfrac{b-a}{n}$.
   Endpoints of the subintervals are $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.
2) Choose a sample point x_i^* in the subinterval $[x_{i-1}, x_i]$, usually left or right endpoint of the subinterval, or the midpoint.
3) Area of rectangle whose height is $f(x_i^*)$ and width is Δx is $f(x_i^*) \cdot \Delta x$, so total area of rectangles is $\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x$.
4) As a greater number of subintervals ought to give a better approximation of area, we get the area under the curve by taking the limit of the approximations:
   $A = \displaystyle\lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i^*)\Delta x \right)$.

Example. The speedometer of a car had readings listed below. Approximate the distance traveled during those 10 minutes.

t (min)	2	4	6	8	10
$v(t)$ (mph)	20	25	15	30	35

Conclusion: distance traveled = area under the curve $v(t)$

(More precisely: displacement = integral under the curve $v(t)$, to allow for negative values of $v(t)$.)

5.2 The Definite Integral

The method of section 5.1 can be applied to any function f over an inteval $[a, b]$.

Definition of the definite integral.

1) Subdivide the interval $[a, b]$ into equal-length subintervals of length $\Delta x = \dfrac{b-a}{n}$.
Endpoints of the subintervals are $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.
2) Choose a sample point x_i^* in the subinterval $[x_{i-1}, x_i]$.
3) Form the sum $\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x$, called a *Riemann sum* for f over the interval $[a, b]$.
4) If the limit $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists, we say f *is integrable* and

define the *definite integral of f from a to b* to be $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i^*)\Delta x \right)$.

Example. What does the Riemann sum represent for the picture above?

As $n \to \infty$, the sum of areas of rectangles above the x-axis approaches the area under the curve and above the x-axis, and the sum of areas of rectangles below the x-axis approaches the area above the curve and below the x-axis.

Therefore, the definite integral represents "signed area" — the area between the curve and the x-axis, where the pieces above the x-axis count as positive, and pieces below the x-axis count as negative.

Use the "signed area" interpretation to compute the folowing integrals.

Example. $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\,dx =$

Example. $\displaystyle\int_{-1}^{2} 1-x\,dx =$

Example. $\displaystyle\int_{-1}^{2} |1-x|\,dx =$

Example. $\displaystyle\int_{a}^{b} c\,dx =$

Note. $\int_a^b f(x)\,dx$ is a number (not a function). The dx does not have any special meaning, but serves as a right parenthesis, marking the end of the function that is being integrated, as well as identifying the variable when there are several letters in the function.

Properties of the definite integral.

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx \qquad\qquad \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Example. If $\int_1^7 f = 3$ and $\int_1^4 f = -2$, how much is $\int_4^7 f$?

More properties of the definite integral.

If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

In particular, if $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.

If $m \leq f(x) \leq M$ on $[a, b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

Picture proofs.

Example. Estimate $\int_0^{\frac{\pi}{2}} \sin x\,dx$.

Definition. If $b < a$, we define (or derive from existing definition, using $\Delta x = \frac{a-b}{n}$):

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

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5.3 Evaluating Definite Integrals

Evaluation Theorem (Fundamental Theorem of Calculus). If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Notation. $F(b) - F(a)$ is written as $F(x)|_a^b$.

Example. $\displaystyle\int_0^2 x^2\,dx =$

Example. $\displaystyle\int_1^2 \frac{x^6 - x^2}{x^3}\,dx =$

Example. $\displaystyle\int_0^{\ln 5} e^{2x}\,dx =$

Example. Find the area under the curve $y = \sin x$ from $x = 0$ to $x = \dfrac{\pi}{2}$.

Example. $\displaystyle\int_{-1}^2 \frac{1}{x^2}\,dx =$

Net Change Theorem.

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

$$\begin{matrix}\text{integral of rate} \\ \text{of change of } F\end{matrix} = \begin{matrix}\text{net change of } F \\ \text{from } a \text{ to } b\end{matrix}$$

Example. At $t = 0$, there are 5 liters of water in a tank, and water is being added at rate $3 - \frac{1}{2}t$ liters per minute at time t minutes.

a) How much water was added from $t = 0$ minutes to $t = 4$ minutes?
b) How much water is in the tank at $t = 4$ minutes?
c) How much water is in the tank at $t = 12$ minutes?

Example. The velocity of an object is given by $v(t) = 9 - t^2$. By how much has it moved from $t = 2$ to $t = 4$?

Note that, since $v(t)$ is derivative of position, the $\int_2^4 v(t)\,dt$ represents change in position, or displacement. If we wanted *distance traveled*, we would compute $\int_2^4 |v(t)|\,dt$.

Definition. Because of the close connection between the definite integral and the antiderivative, we introduce notation $\displaystyle\int f(x)\,dx$ to denote the antiderivative of f.

Example. $\displaystyle\int \cos x\,dx =$

Example. $\displaystyle\int ax + b\,dx =$

Example. $\displaystyle\int \sec\theta\tan\theta\,d\theta =$

5.5 The Substitution Rule

Example. Find $\int ((\ln x)^2 + 3\ln x + 1)\frac{1}{x}\,dx$.

Has form $\int f(g(x))g'(x)\,dx$, which looks like the chain rule $F'(g(x))g'(x)$.

If we can find a function $F(x)$ such that $F'(x) = f(x)$, then the problem becomes

$$\int f(g(x))g'(x)\,dx = \int F'(g(x))g'(x)\,dx = [\text{recognize chain rule}] = F(g(x))$$

which solves it, and all we needed was the antiderivative of f.

Thus, to integrate the form $\int f(g(x))g'(x)\,dx$, we just need to know the antiderivative of f.

The substitution rule formalizes this process:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \qquad \text{once done, substitute back } g(x) \text{ for } u$$

(we "substitute" $g(x)$ with u, and $g'(x)\,dx$ with du)

(The substitution rule is a sort of reverse to the chain rule.)

Example. $\int ((\ln x)^2 + 3\ln x + 1)\frac{1}{x}\,dx =$

Example. $\int \sin^4 x \cos x \, dx =$

Example. $\int (6x^2+2)(x^3+x)^{10} \, dx =$

Example. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx =$

The substitution rule for definite integrals.

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

always change the bounds
and never go back to x

Example. $\int_2^3 \frac{2x+1}{(x^2+x-3)^3} \, dx =$

Example. $\int_{-2}^{2} x \ln(x^2+3)\, dx =$

Example. $\int_{0}^{3} x\sqrt{9-x^2}\, dx =$

Example. Sometimes substitution helps, even though the integral is not obviously in form $\int f(g(x))g'(x)\, dx$.

$$\int \frac{x}{\sqrt{x-1}}\, dx =$$