Calculus 1 — Lecture notes
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4.1 Maximum and Minimum Values

Definition. Let f be defined on domain D and let c be in D. We say:

– $f(c)$ *is the absolute maximum value* of f on D if $f(c) \geq f(x)$ for all x in D.
– $f(c)$ *is the absolute minimum value* of f on D if $f(c) \leq f(x)$ for all x in D.

Another way to say it is that f *has an absolute maximum or minimum at* c, if we want to emphasize at which point the function is greatest or smallest.

Definition. Let f be defined on domain D and let c be in D.

– $f(c)$ *is a local maximum value* of f if $f(c) \geq f(x)$ when x is near c.
– $f(c)$ *is a local minimum value* of f if $f(c) \leq f(x)$ when x is near c.
– "near c" means for all x in some open interval around c

We can also say that f *has a local maximum or minimum at* c, if we want to emphasize at which point the function is greatest or smallest among the points near it.

Absolute or local minimum or maximum values are also referred to as absolute or local *extremes*.

Theorem. If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some points c and d in $[a, b]$.

Proof. Seemingly visually obvious, but not so easy to prove precisely.

Note. If f is not defined on a closed interval, or it is not continuous, then f need not attain an absolute maximum or minimum values.

How to find where a function has extremes?

Fermat's Theorem. If f has a local maximum or minimum at c, and $f'(c)$ exists, then $f'(c) = 0$.

Note. The theorem does *not* say: if $f'(c) = 0$, then f has a local minimum or maximum at c.

Definition. A point c is called a critical point of a function f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Note. Fermat's theorem says that points at which a function has a local extreme are among its critical points.

Example (The Closed Interval Method). Find the absolute maximum and minimum values of the function $f(x) = x^3 - 5x^2 + 3x - 1$ on the interval $[1, 4]$.

Because this is a continuous function defined on a closed interval, both extreme values exist. The function will attain them at either:

– endpoints, or

– interior points, in which case they would be local extremes, which are at critical points.

Therefore, we look for absolute maximum and minimum values at endpoints and critical points.

4.2 The Mean Value Theorem

Rolle's Theorem. Suppose that

1. f is continuous on the interval $[a, b]$.
2. f is differentiable on the interval (a, b).
3. $f(a) = f(b)$

Then there exists a c in (a, b) such that $f'(c) = 0$.

Proof.

Example. Show that the equation $x^3 - x^2 + 5x - 1 = 0$ has exactly one root.

The previous example uses a proof technique called a *contradiction*. Along with our assumptions, we also assume that the opposite of what we are trying to show is true, and arrive at an absurd statement. This tells us that the opposite cannot be true, so the original statement is true.

Example. Use a contradiction to prove there is no largest integer.

The Mean Value Theorem. Suppose that

1. f is continuous on the interval $[a, b]$.
2. f is differentiable on the interval (a, b).

Then there exists a c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof uses Rolle's theorem.

Example. Verify the Mean Value Theorem for the function $f(x) = 2x^3 + x^2 - x - 1$ on the interval $[0, 2]$.

Another interpretation of the Mean Value Theorem:

$f'(c) = \frac{f(b) - f(a)}{b - a}$ says instantaneous rate of change at some c in (a, b) = average rate of change of f over $[a, b]$

Example. A driver entered a 170 mile long toll road at 1PM and exited at 3PM. The speed limit is 70 mph, and the driver got a speeding ticket at the exit. Why?

Example. Let f be a continuous and differentiable function such that $f(1) = 2$ and $3 \leq f'(x) \leq 4$ when x is in $[1, 5]$. Show that $14 \leq f(5) \leq 18$.

Theorem. If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).

Proof.

Corollary. If $f'(x) = g'(x)$ for all x in (a, b), then $f(x) = g(x) + C$ on the interval (a, b), where C is a constant.

Proof.

4.3 Derivatives and Shapes of Graphs

The Increasing / Decreasing Test.

(a) If $f'(x) > 0$ on an interval then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval then f is decreasing on that interval.

"Proof."

Example. Let $f(x) = 2x^3 + 3x^2 - 12x - 7$. Find the intervals where the function is increasing or decreasing and identify local extremes.

The First Derivative Test. Suppose c is a critical point for the function f.

(a) If f' changes sign from $+$ to $-$ at c, f has a local maximum at c.

(b) If f' changes sign from $-$ to $+$ at c, f has a local minimum at c.

(c) If f' does not change sign at c, f does not have a local extreme at c.

When drawing graphs, we would like to include more information about the shape of the graph.

Definition.

A function is *concave upward* if	A function is *concave downward* if
the slopes of tangent lines are increasing	the slopes of tangent lines are decreasing
Alternatively: the graph of the function lies above its tangent lines	*Alternatively:* the graph of the function lies below its tangent lines

Definition. If f is continuous and changes concavity at P, P is called an *inflection point*.

Concavity Test.

(a) If $f''(x) > 0$ on an interval then f is concave upward on that interval.

(b) If $f''(x) < 0$ on an interval then f is concave downward on that interval.

Proof.

Example. Let $f(x) = 2x^3 + 3x^2 - 12x - 7$ (function from previous example). Determine the intervals of concavity of f and use them to draw a more accurate graph of f.

Second Derivative Test. Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Proof.

Summary.

$$f \begin{array}{l}\text{increasing}\\ \text{decreasing}\end{array} \Longleftrightarrow f' \begin{array}{l}> 0\\ < 0\end{array} \qquad\qquad f \begin{array}{l}\text{concave up}\\ \text{concave down}\end{array} \Longleftrightarrow f' \begin{array}{l}\text{increasing}\\ \text{decreasing}\end{array} \Longleftrightarrow f'' \begin{array}{l}> 0\\ < 0\end{array}$$

Example. Given the graph of f, draw the graphs of f' and f''.

4.5 Optimization Problems

Example. Find the point on the line $y = 5 - 3x$ that is closest to the point $(-1, 3)$.

Example. Find the area of the largest rectangle that can be inscribed in a right triangle with sides 3ft and 4ft if two sides of the rectangle lie along the sides of the triangle.

Example. A farmer wants to fence a rectangular area of 1.5km^2 and then divide it in half with a fence parallel to a side of the rectangle. How can he do this so as to minimize the cost of the fence?

Methods of attack of these problems:

1) Draw the @!♣#♢$ picture! Large, clear with indicated quantities.

2) Which quantities are variable? Draw several versions of the picture that reflect the various possibilities.

3) Write the function whose minimum or maximum you are seeking. Express all the variables in terms of one variable. Decide on the interval where you are seeking.

4) Use calculus to find where the function has an extreme. This involves critical points.

5) Using first or second derivative tests, or the closed interval method, ensure that a critical point is a local minimum or maximum. Find the minimum or maximum value.

4.7 Antiderivatives

We consider the problem of "inverse" differentiation: given a function $f(x)$, find a function $F(x)$ such that $F'(x) = f(x)$. $F(x)$ is called the *antiderivative* of $f(x)$.

Example. Find antiderivatives of $f(x) = x^3$.

Theorem. If $F(x)$ is an antiderivative of $f(x)$ on an interval I, then every other antiderivative of $f(x)$ on I has form $F(x) + C$, where C is a constant.

Proof.

Some antiderivatives by guessing

$f(x)$	$F(x)$
1	x
x	$\dfrac{x^2}{2}$
x^n	$\dfrac{x^{n+1}}{n+1}$

$f(x)$	$F(x)$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
a^x	$\dfrac{a^x}{\ln a}$

$f(x)$	$F(x)$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\tan x$	$\ln\lvert\sec x\rvert$

$f(x)$	$F(x)$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$
$\dfrac{1}{1+x^2}$	$\arctan x$

Notes.

Example. Suppose $F' = f$ and $G' = g$, that is, F and G are antiderivatives of f and g. Then $F \pm G$ is an antiderivative of $f \pm g$ and cF is an antiderivative of cf.

Important! There is no such rule for products or quotients, because the rules for derivatives of products and quotients are not so simple.

Example. Find f if f' is the given function.

$f'(x) = 3x^4 - 5\sqrt{x} + \dfrac{1}{x^7}$

Example. Find f if f' is the given function.

$f'(x) = \dfrac{x^2 + 1}{x}$

Example. Find f if f' is the given function.

$f'(x) = e^{3x}$

$f'(x) = e^{7x+5}$

$f'(x) = \sin\left(3x + \dfrac{\pi}{2}\right)$

Note. Above technique works only if the inside function is linear.

$f'(x) = e^{\sin x}$ but $f(x) \neq \dfrac{e^{\sin x}}{\cos x}$

Example (Initial Value Problem). Find a function f such that

$f'(x) = \sqrt[3]{x^2} - \dfrac{1}{\sqrt{x}}$ and $f(1) = 3$

Example. A ball is thrown upward so that at time $t = 1$ its height is 8 meters and its velocity is 5 meters per second. Find the position of the ball at time t.

Example. The above problem can be solved in general: find the position of the object released at time $t = 0$ from height s_0 with velocity v_0.