

Recall that the function $f(x) = a^x$, $a > 0$, $a \neq 1$ is called an exponential function.

Graph:

From the graphs we can see the most important facts about exponential functions.

Continuity:

Domain = Range =

$$\lim_{x \rightarrow \infty} a^x = \qquad \lim_{x \rightarrow -\infty} a^x =$$

Using above facts, we can find limits involving a^x :

Example. $\lim_{x \rightarrow 3} 5^{\frac{x^2 - 4x + 3}{x - 3}} =$

Example. $\lim_{x \rightarrow 4^+} e^{\frac{2}{8 - 2x}} =$

Example. $\lim_{x \rightarrow \infty} \frac{3^x - 1}{3^{2x} + 5 \cdot 3^x - 3} =$

The number e can be defined in several ways, here are two:

1) $e = \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} \approx$

| x | $(1 + x)^{\frac{1}{x}}$ |
|-----------|-------------------------|
| 0.1 | |
| 0.01 | |
| 0.001 | |
| 10^{-4} | |
| 10^{-5} | |
| 10^{-6} | |

2) Let $m_a =$ slope of tangent line to graph of $y = a^x$ at $x = 0$. It can be numerically found that

$$m_2 < 1 \text{ and } m_3 > 1$$

Since m_a varies continuously with a , by the Intermediate Value Theorem there is a number a such that $m_a = 1$.

In this approach, we can define e as the number such that the graph of $y = e^x$ has a tangent line at $x = 0$ whose slope is 1.

Definition. A function is *one-to-one* if it sends different x 's to different y 's, that is

$$\text{if } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2)$$

In other words, it never happens that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Thus, no two points on the graph with different x -coordinates have the same y -coordinate. This is the idea of the:

Horizontal line test. A function is one-to-one if and only if no horizontal line intersects it more than once.

Example. Are the following graphs of one-to-one functions?

One-to-one functions are important because they are the only functions that have inverses: For every y in the range of f , we can define:

$$f^{-1}(y) = \text{the } x \text{ such that } f(x) = y$$

Example. The function $f(x) = x^2$ is not one-to-one, but we have learned that its inverse is \sqrt{x} . What gives?

In general, functions that are not one-to-one, like $\sqrt{\quad}$, \sin , \cos , \tan ,... are turned into one-to-one functions in the same way, by *restricting the domain*.

Inverse functions satisfy: $f^{-1}(f(x)) = x$ $f(f^{-1}(x)) = x$

The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

How to find the derivative of f^{-1} :

Theorem. If f is a one-to-one differentiable function then its inverse f^{-1} is differentiable, and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Example. Use the theorem to find the derivative of $\sqrt[3]{x}$.

Example. Let $f(x) = 2x + \cos x$. Use the theorem to find $(f^{-1})'(1)$.

Definition. The exponential function $f(x) = a^x$ is one-to-one, so has an inverse called the *logarithmic function*: $f^{-1}(x) = \log_a x$.

Note. Think of $\log_a x$ as the answer to the question $a^? = x$, in other words,

$$y = \log_a x \text{ is the same as saying } a^y = x$$

Properties of logarithmic functions

$$\log_a a^x = x \qquad a^{\log_a x} = x$$

are just $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for $f(x) = a^x$ and $f^{-1}(x) = \log_a x$.

| Property | Related exponential property | |
|--|------------------------------|---|
| $\log_a(xy) = \log_a x + \log_a y$ | $a^{u+v} = a^u \cdot a^v$ | Change of base formula: $\log_b x = \frac{\log_a x}{\log_a b}$ |
| $\log_a \frac{x}{y} = \log_a x - \log_a y$ | $a^{u-v} = \frac{a^u}{a^v}$ | |
| $\log_a x^r = r \log_a M$ | $(a^u)^v = a^{uv}$ | |

Special bases: $a = e$, we write $\log_e x = \ln x$
 $a = 10$, we write $\log_{10} x = \log x$

From the graph of a^x , $a > 1$, we get the graph of $\log_a x$ and can see all the important facts about it:

Domain =

Range =

$$\lim_{x \rightarrow \infty} \log_a x =$$

$$\lim_{x \rightarrow 0^+} \log_a x =$$

Example. $\lim_{x \rightarrow 0^+} \log_2(\sin x) =$

Derivative of the exponential function $f(x) = a^x$

Theorem. $\frac{d}{dx} e^x = e^x$ $\frac{d}{dx} a^x = \ln a \cdot a^x$

Example. $\frac{d}{dx} (\sqrt{x}e^x) =$

Example. $\frac{d}{dx} e^{\cos x} =$

Example. $\frac{d}{dx} (x^2 + 3x)2^{4x} =$

Example. $\frac{d}{dx} \frac{x}{e^x} =$

Derivative of the logarithmic function $\log_a x$

Set $f^{-1}(x) = \log_a x$.

Theorem. $\frac{d}{dx} \ln x = \frac{1}{x}$ $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$

Example. $\frac{d}{dx} \ln \sqrt{x} =$

Example. $\frac{d}{dx} \ln \left(\frac{x+1}{x-1} \right) =$

Example. $\frac{d}{dx} \ln(\cos x) =$

Example. $\frac{d}{dx} (x^2 - 7x) \log_3 x =$

Example. $\frac{d}{dx} \log_5(\tan x) =$

Example. Find the derivative of $y = \frac{e^{3x}\sqrt{x^2+1}}{(x^3+17)^4}$. This would be hard using the quotient rule (which would include a product rule for the derivative of the numerator), but we can simplify work using the trick of “logarithmic differentiation.”

Example. Use logarithmic differentiation to find the derivative of $y = x^x$. Same method can be used to find the derivative of any function of form $f(x)^{g(x)}$.

The functions \sin , \cos and \tan are not one-to-one functions, so in order for them to have an inverse, we first make them one-to-one by restricting the domain. The functions \arcsin , \arccos and \arctan are inverses of the functions \sin , \cos and \tan restricted as follows.

We can say: $\arcsin x$ is the angle θ whose sine is x and falls in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\arccos x$ is the angle θ whose cosine is x and falls in $[0, \pi]$

$\arctan x$ is the angle θ whose tangent is x and falls in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Derivatives of inverse trigonometric functions.

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

We justify the derivatives of inverse trigonometric functions.

Example. $\frac{d}{dx} \arctan(7x) =$

Example. $\frac{d}{dx} (\arctan x)^2 =$

Example. $\frac{d}{dx} (x \arcsin x + \sqrt{1-x^2}) =$

When computing limits, the difficult ones are always an indeterminate form:

$$\infty - \infty \quad 0 \cdot \infty \quad \frac{\infty}{\infty} \quad \frac{0}{0}$$

The rule below helps us find some of them.

Theorem (L'Hospital's Rule). Suppose f and g are differentiable near a and $g'(x) \neq 0$ near a . If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}, \text{ if the latter exists, or is } \pm \infty$$

The rule also holds when $\lim_{x \rightarrow a} f(x) = \pm \infty$ and $\lim_{x \rightarrow a} g(x) = \pm \infty$, or the limit is one-sided.

Note. The rule helps with forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Note that this is *not* the quotient rule for derivatives, it is a statement about limits that uses derivatives.

Example. $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos^2 x}{\sin x - 1} =$

Example. $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} =$

Example. $\lim_{x \rightarrow 0^+} x \ln x =$

Exponential indeterminate forms are: 0^0 , ∞^0 , 1^∞ .

Example. $\lim_{x \rightarrow 0^+} x^{\sqrt{x}} =$

Example. $\lim_{x \rightarrow \infty} \left(\frac{x}{x+1} \right)^x =$

Example. $\lim_{x \rightarrow \infty} \frac{x^3}{e^x} =$

Similarly, $\lim_{x \rightarrow \infty} \frac{x^c}{e^x} = 0$ for any $c > 0$, that is, e^x grows faster than any x^c , $c > 0$, which is interesting for large positive numbers c .

Example. $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} =$

Similarly, $\lim_{x \rightarrow \infty} \frac{\ln x}{x^c} = 0$ for any $c > 0$, that is, $\ln x$ grows slower than any x^c , $c > 0$, which is interesting for small positive numbers c .