Introduction to Topology — Exam 1 **MAT 516/616 , Spring 2022 — D. Ivanšić**

Name: ______________________________

Show all your work!

Do all the theory problems. Then do at least five problems, one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Define a limit point of A.

Theory 2. (3pts) State the theorem that describes what open sets are in $(\mathbf{R}, \mathcal{U})$.

Theory 3. (3pts) Let d be a metric on a set X. Define the metric topology on X.

Type A problems (5pts each)

A1. Let $X = \mathbf{N}$ (natural numbers) and $\mathcal{T} = \{U \subseteq \mathbf{N} \mid U = \emptyset \text{ or } U \text{ contains at least one even number}\}$. Is $\mathcal{T}$ a topology?

A2. Let $A = (0, 2) \cup (4, \infty)$ be a subset of the topological space $(\mathbf{R}, \mathcal{C})$. Find $\operatorname{Bd} A$ and justify, possibly with pictures.

A3. Let $A = \left\{ (-1)^n \frac{1}{n} \mid n \in \mathbf{N} \right\}$. Determine $\operatorname{Cl} A$ and justify, possibly with pictures.

A4. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ be a continuous function between topological spaces and let $B \subseteq Y$. Show that $f^{-1}(\operatorname{Int} B) \subseteq \operatorname{Int} f^{-1}(B)$.

A5. Let $f : (\mathbf{R}, \mathcal{H}) \to (\mathbf{R}, \mathcal{U})$ be the function defined by $f(x) = x^2$. Show that f is continuous.

A6. Let $\mathcal{B} = \{U \subseteq \mathbf{R} \mid U = [0, 1] \text{ or } U = [0, 1] \cup (a, b), \text{ where } a, b \in \mathbf{R} \text{ and } a < b\}$. Show that $\mathcal{B}$ is a base for a topology on $\mathbf{R}$. State, without proof, the open sets of this topology.

Type B problems (8pts each)

B1. Let $f : \mathbf{R} \to \mathbf{R}$ be the function given below. Determine whether f is
a) $\mathcal{U}$–$\mathcal{U}$ continuous b) $\mathcal{U}$-$\mathcal{C}$ continuous

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ x + 1, & \text{if } x < 0. \end{cases}$$

B2. Let $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, X, \{a\}, \{a, b\}\}$, $A = \{b, c\}$. Determine $\operatorname{Int} A$, A' and $\operatorname{Cl} A$ and justify.

B3. Consider the topological space $(\mathbf{R}, \mathcal{C})$. Show that $A \subseteq \mathbf{R}$ is dense in $\mathbf{R}$ if and only if A is not bounded above.

B4. Let A, B be subsets of a topological space $(X, \mathcal{T})$. Show that $\operatorname{Cl}(A \cup B) = \operatorname{Cl} A \cup \operatorname{Cl} B$. (Hint: don't do anything complicated. Use properties of closure.)

B5. Show that the collection $\mathcal{B} = \{(a, b) \mid a, b \in \mathbf{R}, a$ rational and b irrational$\}$ is a base for $(\mathbf{R}, \mathcal{U})$.

B6. Show that the function d defined below is a metric on $\mathbf{Z}$ (integers).

$$d(m,n) = \begin{cases} 0, & \text{if } m = n \\ |m| + |n| & \text{if } m \neq n \end{cases}$$

Type C problems (12pts each)

C1. Let $\mathcal{B} = \{[a, b) \mid a, b \in \mathbf{R}, a$ rational and b irrational$\}$.
a) Show that the collection $\mathcal{B}$ is a base for a topology on $\mathbf{R}$.
b) Show $\mathcal{B}$ is not a base for the topology $\mathcal{H}$.

C2. Let $\{A_\alpha \mid \alpha \in \Lambda\}$ be a collection of sets in a topological space X. Show that

$$\bigcup_{\alpha \in \Lambda} \operatorname{Cl}(A_\alpha) \subseteq \operatorname{Cl}\left(\bigcup_{\alpha \in \Lambda} A_\alpha\right).$$

Give an example where the two sets are not equal and justify.

Introduction to Topology — Exam 2 MAT 516/616 , Spring 2022 — D. Ivanšić

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Show all your work!

Do all the theory problems. Then do at least five problems, one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Let $(X_1, \mathcal{T}_1), \dots, (X_n, \mathcal{T}_n)$ be topological spaces. Define the product topology on $X_1 \times \cdots \times X_n$.

Theory 2. (3pts) Define a homeomorphism.

Theory 3. (3pts) Let X be a topological space and $A \subset X$. State the theorem that gives a criterion for when a subset $B \subseteq A$ is closed in the relative topology.

Type A problems (5pts each)

A1. Let $A = [-2, 2]$ be a subspace of $(\mathbf{R}, \mathcal{U})$. Which of the subsets of A are open in the relative topology: $(-2, 0)$, $(0, 2]$, $(-1, 1]$? Prove your answers.

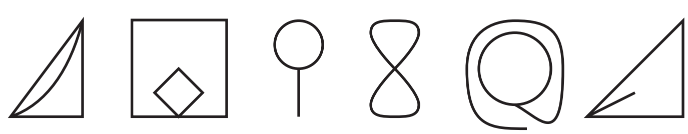
A2. Let X be a topological space with base $\mathcal{B}$, and let $A \subseteq X$. Show that the collection $\mathcal{B}' = \{B \cap A \mid B \in \mathcal{B}\}$ is a base for the relative topology on A.

A3. Let $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \{\emptyset, \{a\}, \{a, c, d\}, \{a, b, c, d\}\}$. Let $A = \{a, b, c\}$. Find $\operatorname{Cl}\{c\}$ and $\operatorname{Cl}_A\{c\}$ and justify your answer.

A4. Let $f : X \to Y$ be a continuous function and $B \subseteq Y$. Show that $f^{-1}(\operatorname{Int} B) \subseteq \operatorname{Int} f^{-1}(B)$.

A5. Consider the product space $(\mathbf{R}, \mathcal{U}) \times (\mathbf{R}, \mathcal{C})$. Draw a subset of $\mathbf{R} \times \mathbf{R}$ that is open in the product topology but is not a base element. (Justify that it is open, but you do not have to justify that it is not a base element.)

A6. Group the subspaces of $\mathbf{R}^2$ into groups of homeomorphic spaces. Show spaces from one pair of groups are not homeomorphic.



Type B problems (8pts each)

B1. Give an example of an increasing function $f : \mathbf{R} \to \mathbf{R}$ that is not continuous as $f : (\mathbf{R}, \mathcal{U}) \to (\mathbf{R}, \mathcal{U})$ but is continuous as $f : (\mathbf{R}, \mathcal{C}) \to (\mathbf{R}, \mathcal{C})$. (Think simple.) Justify.

B2. Let X be a topological space and $A \subset X$. For any subset $C \subseteq A$, show that $\operatorname{Bd}_A C \subseteq A \cap \operatorname{Bd}_X C$.

B3. Let $f, g : \mathbf{R} \to \mathbf{R}$ be continuous functions and let $g(x) \neq 0$ for all $x \in \mathbf{R}$. Show that the function $\frac{f}{g}$ is continuous using known theorems from chapter 4.

B4. Let $A \subseteq \mathbf{R}^2$ be the parabola $y = x^2$ with the relative topology. Show that A is homeomorphic to $\mathbf{R}$.

B5. Let $\mathcal{D}$ be the discrete topology. Prove with pictures that the multiplication function

$$m : (\mathbf{R}, \mathcal{U}) \times (\mathbf{R}, \mathcal{D}) \to (\mathbf{R}, \mathcal{U})$$

is continuous. (Hint: what are basic open sets in $(\mathbf{R}, \mathcal{U}) \times (\mathbf{R}, \mathcal{D})$ like?)

B6. Let $X = \{0, 1, 2, \dots\}$ and $Y = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbf{N}\right\}$, and let X and Y have the subspace topologies coming from $(\mathbf{R}, \mathcal{U})$. Show that X is not homeomorphic to Y.

Type C problems (12pts each)

C1. Let X and Y be topological spaces, $A \subseteq X$, $B \subseteq Y$. Show that

$$\mathrm{Bd}(A \times B) = (\mathrm{Bd}\, A \times \mathrm{Cl}\, B) \cup (\mathrm{Cl}\, A \times \mathrm{Bd}\, B).$$

Illustrate for $X, Y = \mathbf{R}$, $A = (2, 4)$, $B = (1, 3)$. (Hint: you will not need to go into definitions, just use established properties.)

C2. Show that any polynomial in two variables $P : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is continuous using known theorems from chapter 4. (An example of a polynomial in two variables is $P(x, y) = x^3 + 6x^2y - 5xy + y$.)

Introduction to Topology — Exam 3 **MAT 516/616 , Spring 2022 — D. Ivanšić**	**Name:** ______________________ *Show all your work!*

Do all the theory problems. Then do at least five problems, one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define when a topological space X is compact.

Theory 2. (3pts) Define a fixed point and the fixed point property.

Theory 3. (3pts) State the theorem about connectedness of $X \times Y$.

Type A problems (5pts each)

A1. Let $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \{\emptyset, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}$, and let $A = \{b, c\}$. Is X connected? Is A connected? Prove your answers.

A2. Show that any subset of $(\mathbf{R}, \mathcal{C})$ is connected.

A3. True or false? Let $f : X \to Y$ be a continuous function. If $B \subseteq Y$ is connected, then $f^{-1}(B)$ is connected. Justify your answer.

A4. Let the natural numbers $\mathbf{N}$ have the finite-complement topology $\mathcal{T} = \{U \subseteq \mathbf{N} \mid U^c \text{ is finite or } U = \emptyset\}$. Is $(\mathbf{N}, \mathcal{T})$ Hausdorff?

A5. Show that $(1, 4)$ is not a compact subset of $(\mathbf{R}, \mathcal{C})$.

A6. Give an example of a disconnected compact subset of $(\mathbf{R}, \mathcal{U})$ and justify why it is disconnected and compact.

Type B problems (8pts each)

B1. Prove the generalized Intermediate Value Theorem: let X be a connected space, $f : X \to \mathbf{R}$ a continuous function and let $a, b \in X$. If N is any number between $f(a)$ and $f(b)$, then there exists a $c \in X$ such that $f(c) = N$.

B2. Let $A \subseteq \mathbf{R} \times \mathbf{R}$ be the set consisting of the graph of $f(x) = \frac{1}{x} \sin \frac{1}{x}$ for $x > 0$ and the y-axis. Show that A is connected. (Use known theorems and properties of connectedness rather than the definition.)

B3. Let $f : (\mathbf{R}, \mathcal{H}) \to (\mathbf{R}, \mathcal{H})$ be a continuous function. Does the Intermediate Value Theorem hold in this case? Justify.

B4. Show that $[1, 4]$ is not a compact subset of $(\mathbf{R}, \mathcal{H})$. (Hint: there is an infinite open cover consisting of disjoint base elements of $\mathcal{H}$ that utilizes the point 3, for example.)

B5. Let X have the discrete topology. Construct and prove the statement: X is compact if and only if X is ______.

B6. Let X be a space with the finite-complement topology. Show that any subset $A \subseteq X$ is compact.

B7. Let Y be a Hausdorff space, and let $f : X \to Y$ be a continuous function, and let $X \times Y$ have the product topology. Define $G \subseteq X \times Y$ to be the "graph" of f:

$$G = \{(x, f(x)) \in X \times Y \mid x \in X\}.$$

Show that G is a closed subset of $X \times Y$.

Type C problems (12pts each)

C1. Show that $(\mathbf{R} \times \mathbf{R}) - (\mathbf{Q} \times \mathbf{Q})$ is path-connected (and hence connected). Pictures will suffice as an argument.

C2. Prove that a subset A of $(\mathbf{R}, \mathcal{C})$ is compact if and only if $\inf A \in A$. (Note that this means $\inf A$ is a real number.)