Introduction to Topology — Final Exam **MAT 516/616 , Fall 2013 — D. Ivanšić**	**Name:** ______________ *Show all your work!*

Do all the theory problems. Then do at least five problems, one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Define $\operatorname{Cl} A$.

Theory 2. (3pts) Define the product topology.

Theory 3. (3pts) Define when a topological space X is compact.

Type A problems (5pts each)

A1. Let $(X, \mathcal{T})$ be a topological space. Show that a subset $A \subseteq X$ is dense if and only if for every open set U, $U \cap A \neq \emptyset$.

A2. Let X be any set with three elements or more and $\mathcal{B}$ be the collection of all two-element subsets of X. Show that $\mathcal{B}$ is not a base for any topology.

A3. Let X be a topological space and let $A \subset X$ be a closed set. Show that $F \subseteq A$ is closed in A if and only if F is closed in X.

A4. Let X be a topological space and define $f : X \times X \to X \times X$ as $f(x, y) = (y, x)$. Show that f is a homeomorphism. What is its inverse?

A5. Show that any subset of $(\mathbf{R}, \mathcal{H})$ with at least two elements is disconnected.

A6. Show that $(1, 4)$ is not a compact subset of $(\mathbf{R}, \mathcal{H})$.

A7. Show that $[3, \infty)$ is a compact subset of $(\mathbf{R}, \mathcal{C})$.

Type B problems (8pts each)

B1. Let $f : \mathbf{R} \to \mathbf{R}$ be the function given below. Determine whether f is
a) $\mathcal{U}$–$\mathcal{U}$ continuous b) $\mathcal{C}$-$\mathcal{C}$ continuous

$$f(x) = \begin{cases} x, & \text{if } x > 0 \\ x - 1, & \text{if } x \leq 0. \end{cases}$$

B2. Show that the collection $\mathcal{B} = \{B_{\frac{1}{n}}(x) \mid x \in \mathbf{R}, n \in \mathbf{N}\}$ is a base for $(\mathbf{R}, \mathcal{U})$, where the balls are defined using the usual metric $d(x, y) = |x - y|$.

B3. Let X have the discrete topology. Show that the product topology on $X \times X$ is the discrete topology (i.e. every subset of $X \times X$ is open in the product topology).

B4. Let $A \subseteq \mathbf{R} \times \mathbf{R}$ be the set $A = ([0,1) \times \mathbf{R}) \cup \{(1,n) \mid n \in \mathbf{Z}\}$. Is A connected? Prove your answer.

B5. Show that the punctured plane $\mathbf{R} \times \mathbf{R} - \{(0,0)\}$ is a connected space, with the topology relative to the usual product topology on $\mathbf{R} \times \mathbf{R}$. (Use known theorems and properties of connectedness rather than the definition.)

B6. Let X be a Hausdorff space, and let $X \times X$ have the product topology. Define $Z \subseteq X \times X$ to be the set $Z = \{(x,x) \in X \times X \mid x \in X\}$. Show that Z is a closed subset of $X \times X$.

Type C problems (12pts each)

C1. A collection $\mathcal{T}$ of subsets of the set of natural numbers $\mathbf{N}$ is defined as follows: $U \in \mathcal{T}$ provided that for every $m \in U$, all the divisors of m are also in U, where 1 is considered a divisor.

For example: $\{1,2,3,4,6,12\}$, $\{1,2,4,8\}$, $\{1,7\}$, $\{3^k \mid k \geq 0\}$ are in $\mathcal{T}$;
$\{2,5,10\}$, $\{14,28\}$, $\{1,5,6,30\}$ are not in $\mathcal{T}$.

a) Show that $\mathcal{T}$ is a topology on $\mathbf{N}$.
b) Find $\operatorname{Cl} A$, where $A = \{15\}$.

C2. Let $X = \mathbf{R}^2 - \{(0,0)\}$ and $Y = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 > 1\}$. Show that X and Y are homeomorphic. First sketch what the homeomorphism does, then write a formula for it and prove it is a homeomorphism.

C3. Let X be the square $[0,1] \times [0,1]$ with the usual product topology and $f : X \to \mathbf{R}$ a continuous function so that $f(0,0) = -3$ and $f(1,1) = 2$. Show that there is a point (c,d) in $(0,1) \times (0,1)$ (the "interior") such that $f(c,d) = 0$.

C4. Give an example of a compact subset of $(\mathbf{R}, \mathcal{U})$ that is not a closed interval. Then show: if $A \subseteq (\mathbf{R}, \mathcal{U})$ is a compact and connected subset, then A is a closed interval.