Introduction to Topology — Exam 3 **MAT 516/616 , Fall 2013 — D. Ivanšić**

Name: ______________________________

Show all your work!

Do all the theory problems. Then do at least five problems, one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define when a topological space X is connected.

Theory 2. (3pts) Define a Hausdorff space.

Theory 3. (3pts) State the theorem that characterizes compact subsets of $\mathbf{R}$ (Heine-Borel Theorem).

Type A problems (5pts each)

A1. Let $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$, and let $A = \{a, b\}$. Is X connected? Is A connected? Prove your answers.

A2. Show that any subset of $(\mathbf{R}, \mathcal{H})$ with at least two elements is disconnected.

A3. Let $\mathbf{R}$ have the topology $\mathcal{T} = \{U \subseteq \mathbf{R} \mid U = \mathbf{R} \text{ or } [-2, 2] \subseteq U^c\}$. Is $(\mathbf{R}, \mathcal{T})$ connected?

A4. Is the set $\mathbf{Q} \cap [0, 1]$ (with usual topology) compact? Prove your answer.

A5. Use the Intermediate Value Theorem to show there exists a real number c satisfying $c^3 = 5$.

A6. Show that $(1, 4)$ is not a compact subset of $(\mathbf{R}, \mathcal{H})$.

A7. Show that $[3, \infty)$ is a compact subset of $(\mathbf{R}, \mathcal{C})$.

Type B problems (8pts each)

B1. Show that any connected subset A of $(\mathbf{R}, \mathcal{U})$ has the property: if $x < y$ are elements of A, then any $z \in (x, y)$ is also in A.

B2. Let $A \subseteq \mathbf{R} \times \mathbf{R}$ be the set $A = ([0, 1) \times \mathbf{R}) \cup \{(1, n) \mid n \in \mathbf{Z}\}$. Is A connected? Prove your answer.

B3. Let X be a topological space and A a finite subset of X. Show that A is compact.

B4. Show that the punctured plane $\mathbf{R} \times \mathbf{R} - \{(0, 0)\}$ is a connected space, with the topology relative to the usual product topology on $\mathbf{R} \times \mathbf{R}$. (Use known theorems and properties of connectedness rather than the definition.)

B5. Let X be a Hausdorff space, and let $X \times X$ have the product topology. Define $Z \subseteq X \times X$ to be the set $Z = \{(x, x) \in X \times X \mid x \in X\}$. Show that Z is a closed subset of $X \times X$.

Type C problems (12pts each)

C1. Let X be the square $[0,1] \times [0,1]$ with the usual product topology and $f : X \to \mathbf{R}$ a continuous function so that $f(0,0) = -3$ and $f(1,1) = 2$. Show that there is a point (c,d) in $(0,1) \times (0,1)$ (the "interior") such that $f(c,d) = 0$.

C2. Give an example of a compact subset of $(\mathbf{R}, \mathcal{U})$ that is not a closed interval. Then show: if $A \subseteq (\mathbf{R}, \mathcal{U})$ is a compact and connected subset, then A is a closed interval.