Introduction to Topology — Exam 2
MAT 516/616 , Fall 2013 — D. Ivanšić

Name: ______________________

Show all your work!

Do all the theory problems. Then do at least five problems, one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Let X be a topological space and $A \subseteq X$. Define the relative topology on A.

Theory 2. (3pts) Define the neighborhood of point.

Theory 3. (3pts) Let X, $Y_1, \dots, Y_n$ be topological spaces. State the theorem that gives a criterion for when a function $f : X \to Y_1 \times \cdots \times Y_n$ is continuous, where $Y_1 \times \cdots \times Y_n$ has the product topology.

Type A problems (5pts each)

A1. Let $A = (-\infty, 7]$ be a subspace of $(\mathbf{R}, \mathcal{U})$. Which of the subsets of A are open in the relative topology: $(-1, 3)$, $(-\infty, 4]$, $(0, 7]$? Prove your answers.

A2. Let $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$. Let $A = \{a, b, c\}$. Find $\text{Int}\{a, c\}$ and $\text{Int}_A\{a, c\}$ and justify your answer.

A3. Let X be a topological space and let $A \subset X$ be a closed set. Show that $F \subseteq A$ is closed in A if and only if F is closed in X.

A4. Give an example of a function $f : (\mathbf{R}, \mathcal{H}) \to (\mathbf{R}, \mathcal{H})$ that is not continuous. Then give an example of a function that is not open.

A5. Let X be a topological space and define $f : X \times X \to X \times X$ as $f(x, y) = (y, x)$. Show that f is a homeomorphism. What is its inverse?

A6. Group the subspaces of $\mathbf{R}^2$ into groups of homeomorphic spaces. Show spaces from one pair of groups are not homeomorphic.

Type B problems (8pts each)

B1. Let X and Y be topological spaces with bases $\mathcal{B}_1$ and $\mathcal{B}_2$. Show that the collection $\mathcal{B} = \{B_1 \times B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$ is a base for the product space $X \times Y$.

B2. Let $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c, d\}\}$. Find all the homeomorphisms $f : X \to X$ and justify that your list is exhaustive.

B3. Let X have the discrete topology. Show that the product topology on $X \times X$ is the discrete topology (i.e. every subset of $X \times X$ is open in the product topology).

B4. Let X be a topological space, and let $X \times X$ have the product topology. Define $Z \subseteq X \times X$ to be the set $Z = \{(x, x) \in X \times X \mid x \in X\}$.
a) Draw the set Z for the closed interval $X = [1, 2]$.
b) Show that X is homeomorphic to Z (Z has the relative topology).

B5. Let $\mathbf{R}$ have the topology $\mathcal{C}$. Show with a picture that the multiplication function $m : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is not continuous. (Hint: what are basic open sets in $\mathbf{R} \times \mathbf{R}$ like?)

Type C problems (12pts each)

C1. Let X and Y be topological spaces, $A \subseteq X$, $B \subseteq Y$. Show that

$$\mathrm{Bd}(A \times B) = (\mathrm{Bd}\, A \times \mathrm{Cl}\, B) \cup (\mathrm{Cl}\, A \times \mathrm{Bd}\, B).$$

Illustrate for $X, Y = \mathbf{R}$, $A = (2, 4)$, $B = (1, 3)$. (Hint: you will not need to go into definitions, just use established properties.)

C2. Let $X = \mathbf{R}^2 - \{(0, 0)\}$ and $Y = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 > 1\}$. Show that X and Y are homeomorphic. First sketch what the homeomorphism does, then write a formula for it and prove it is a homeomorphism.