Introduction to Topology — Exam 1 **MAT 516/616 , Fall 2013 — D. Ivanšić**

Name: ____________________________

Show all your work!

Do all the theory problems. Then do at least five problems, one of which is of type B or C (two if you are a graduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Let X be a set. Define what a topology on X is.

Theory 2. (3pts) Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Define $\operatorname{Int} A$.

Theory 3. (3pts) Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. State the theorem that reveals the relationship between $\operatorname{Int} A$, $\operatorname{Bd} A$ and $\operatorname{Ext} A$.

Type A problems (5pts each)

A1. Let $X = \{a, b, c\}$. Which of the following collections is a topology on X?

$\mathcal{T}_1 = \{\emptyset, \{a\}, \{a, b\}\}$, $\mathcal{T}_2 = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$, $\mathcal{T}_3 = \{\emptyset, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

A2. Let $A = (-2, 2) \cup \{5\}$ be a subset of the topological space $(\mathbf{R}, \mathcal{H})$. Find A', $\operatorname{Int} A$ and $\operatorname{Bd} A$.

A3. Let $A = \mathbf{Q} \cap [0, 1]$ (all rationals between 0 and 1) be a subset of the topological space $(\mathbf{R}, \mathcal{U})$. Determine $\operatorname{Cl} A$.

A4. Let A be a subset of a the topological space $(X, \mathcal{T})$. Show that $\operatorname{Cl} A = \operatorname{Int} A$ if and only if A is both open and closed. (Hint: don't do anything complicated. Use properties of interior and closure.)

A5. Let $(X, \mathcal{T})$ be a topological space. Show that a subset $A \subseteq X$ is dense if and only if for every open set U, $U \cap A \neq \emptyset$.

A6. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ be the function between topological spaces defined by $f(x) = y_0$, where y_0 is a fixed element of Y. Show that f is continuous.

A7. Let X be any set with three elements or more and $\mathcal{B}$ be the collection of all two-element subsets of X. Show that $\mathcal{B}$ is not a base for any topology.

Type B problems (8pts each)

B1. Let $f : \mathbf{R} \to \mathbf{R}$ be the function given below. Determine whether f is
a) $\mathcal{U}$–$\mathcal{U}$ continuous b) $\mathcal{C}$-$\mathcal{C}$ continuous

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ x - 1, & \text{if } x < 0. \end{cases}$$

B2. Show that the linear function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = mx + b$, $m > 0$ is $\mathcal{C}$–$\mathcal{C}$ continuous. How about when $m = 0$ or $m < 0$?

B3. For every subset A of a topological space X show that $\operatorname{Cl} A = A \cup \operatorname{Bd} A$.

B4. Show that the collection $\mathcal{B} = \{B_{\frac{1}{n}}(x) \mid x \in \mathbf{R}, n \in \mathbf{N}\}$ is a base for $(\mathbf{R},\mathcal{U})$, where the balls are defined using the usual metric $d(x, y) = |x - y|$.

B5. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ be a function between topological spaces and let $\mathcal{B}$ be a base for $\mathcal{S}$. Show that f is continuous if and only if $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.

B6. Show that the function $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + 5|y_1 - y_2|$ is a metric on $\mathbf{R}^2$. Determine what $B_1(0, 0)$ is.

B7. Show that the function d defined below is a metric on $\mathbf{N}$ (natural numbers).

$$d(m, n) = \begin{cases} 0, & \text{if } m = n \\ \frac{1}{2}, & \text{if } m \neq n \text{ and both are even or both are odd} \\ 1 & \text{if one is even and the other is odd.} \end{cases}$$

Type C problems (12pts each)

C1. A collection $\mathcal{T}$ of subsets of the set of natural numbers $\mathbf{N}$ is defined as follows: $U \in \mathcal{T}$ provided that for every $m \in U$, all the divisors of m are also in U, where 1 is considered a divisor.

For example: $\{1, 2, 3, 4, 6, 12\}$, $\{1, 2, 4, 8\}$, $\{1, 7\}$, $\{3^k \mid k \geq 0\}$ are in $\mathcal{T}$;
$\{2, 5, 10\}$, $\{14, 28\}$, $\{1, 5, 6, 30\}$ are not in $\mathcal{T}$.

a) Show that $\mathcal{T}$ is a topology on $\mathbf{N}$.
b) Find $\operatorname{Cl} A$, where $A = \{15\}$.

C2. For any set A in a topological space X, show that $\operatorname{Cl}(\operatorname{Int}(\operatorname{Cl}(\operatorname{Int} A))) = \operatorname{Cl}(\operatorname{Int} A)$. Show also that $\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(\operatorname{Cl} A))) = \operatorname{Int}(\operatorname{Cl} A)$. (Hint: don't do anything complicated. Use properties of interior and closure.)