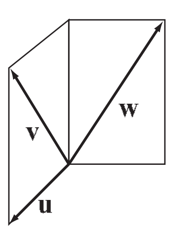
Calculus 3 — Exam 1
MAT 309, Spring 2022 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (11pts) Let $\mathbf{u} = \langle 2, -1, -3 \rangle$ and $\mathbf{v} = \langle 4, 1, 0 \rangle$.
a) Calculate $3\mathbf{u}$, $2\mathbf{u} - 5\mathbf{v}$, and $\mathbf{u} \cdot \mathbf{v}$.
b) Find a vector of length 4 in direction of $\mathbf{u}$.
c) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

2. (10pts) In the picture are two rectangles of width 2 units and height 3 units that are perpendicular to each other.
a) Draw the vectors $\mathbf{u} \times \mathbf{v}$ and $\mathbf{w} \times \mathbf{v}$, accurate length not being important.
b) What is the length of $\mathbf{u} \times \mathbf{w}$?



3. (8pts) Draw the set in
$\mathbf{R}^3$ described by:

$x^2 + y^2 + z^2 < 4,\ z > 1$

4. (13pts) Find the parametric equations of the line that is the intersection of planes $3x + y - z = 2$ and $2x - y + 4z = 1$.

5. (16pts) This problem is about the surface $-\frac{x^2}{9} + \frac{y^2}{9} + \frac{z^2}{4} = 1$.
a) Identify and sketch the intersections of this surface with the coordinate planes.
b) Sketch the surface in 3D, with coordinate system visible.

6. (14pts) The curve $\mathbf{r}(t) = \langle \cos t, \sin t, \sec t \rangle$ is given, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.
a) Sketch the curve in the coordinate system.
b) Find parametric equations of the tangent line to this curve when $t = 0$ and sketch the tangent line.

7. (14pts) Find the length of the curve $\mathbf{r}(t) = \langle t^2 \sin t, t^2 \cos t, 2t \rangle$, $0 \le t \le 1$.

8. (14pts) A squishy is launched from point $(4, 1, 2)$ with initial velocity $\langle -2, 4, 7 \rangle$.
a) Assuming gravity acts in the usual negative $\mathbf{k}$-direction (let $g = 10$), find the vector function $\mathbf{r}(t)$ representing the position of the squishy.
b) When and at which point does the squishy hit the yz-plane?

Bonus (10pts) Find a unit vector that makes an angle of measure $\frac{\pi}{4}$ with $\mathbf{i}$ and an angle of measure $\frac{\pi}{3}$ with $\mathbf{k}$. How many solutions are there?

Calculus 3 — Exam 2
MAT 309, Spring 2022 — D. Ivanšić

Name:

Show all your work!

1. (14pts) Let $f(x, y) = x^2 - y^2$.
a) Sketch the contour map for the function, drawing level curves for levels $k = -1, 0, 1$. Note the domain on the picture.
b) Without computation, draw the directions of $\nabla f(0, 1)$ and $\nabla f(1, \frac{1}{\sqrt{2}})$. Note that these points are on the level curves you drew in b)

2. (14pts) Mice are roaming around a flat board with a sticky surface, whose stickiness is given by the function $f(x, y) = (x^2 + y^2)e^{-3x}$. One mouse, located at point $(0, 1)$ sees another mouse located at point $(-1, 7)$ and starts moving towards it.
a) At first, does the mouse experience an increase or decrease in stickiness?
b) In what direction should the mouse at point $(0, 1)$ move in order to achieve the greatest stickiness decrease, and what is the rate of change of stickiness in that direction?

3. (12pts) Find the equation of the tangent plane to the ellipsoid $\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{9}=10$ at the point $(1, 2, -6)$. Simplify the plane equation to standard form.

4. (16pts) Let $z=\frac{\cos x}{\sin x+\cos y}$, $x=2u-3v^2+6$, $y=u^2-4v-1$. Use the chain rule to find $\frac{\partial z}{\partial v}$ when $u=3$, $v=2$.

5. (14pts) The surface area of a cone of radius r and height h is given by $S = \pi r\sqrt{r^2 + h^2}$ (bottom disk not included). Starting with a cone with radius 6 meters and height 8 meters, use differentials to estimate by how much the surface area changes if the radius increases by 0.1 meters and height decreases 0.3 meters.

6. (10pts) Use implicit differentiation to find $\dfrac{\partial z}{\partial y}$ at the point $(e, 1, e)$, if $x \ln y + y \ln x + z \ln x = e + 1$.

7. (20pts) Find and classify the local extremes for $f(x, y) = y^3 + 3x^2y - 3x^2 - \frac{15}{2}y^2$.

Bonus (10pts) Suppose pollen is distributed in the plane with concentration $C(x, y) = x^2 + 2y^2$. A bee moving in the plane always tries to go in direction of the greatest increase of pollen concentration. Show that it will move along the curve $y = kx^2$ for some k. That is, show that a parametrization $\mathbf{r}(t)$ for this curve satisfies that $\mathbf{r}'(t)$ is always parallel to $\nabla C(\mathbf{r}(t))$.

Calculus 3 — Exam 3 **MAT 309, Spring 2022 — D. Ivanšić**	**Name:** ______________________ *Show all your work!*

1. (16pts) Let D be the region bounded by the curves $y = e^x - 1$, $y = 0$ and $x = 2$.
a) Sketch the region D.
b) Set up $\displaystyle\iint_D \sqrt{e^x - x}\, dA$ as iterated integrals in both orders of integration.
c) Evaluate the double integral using the easier order.

2. (12pts) Let D be the triangle with vertices $(1, 0)$, $(0, 1)$ and $(2, 1)$. Set up $\iint_D \dfrac{x + y}{x^2 + y^2}\, dA$, but do not evaluate the integral. Sketch the region of integration first.

3. (22pts) Use polar coordinates to find the area of the region inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$. Sketch the region of integration first.

4. (18pts) Sketch the region E that is under the paraboloid $z = 4 - x^2 - y^2$, above the xy-plane and in front of plane $x = 1$ (so points of the region satisfy $x \geq 1$). Then write the two iterated triple integrals that stand for $\iiint_E f\,dV$ which end in $dy\,dz\,dx$ and $dx\,dz\,dy$.

5. (16pts) Use spherical coordinates to set up the integral $\iiint_E x^2+y^2\,dV$, if E is the region that is inside the sphere $x^2+y^2+z^2=9$ and between the planes $y=x$ and $y=-x$, the part that intersects the positive x-axis. Simplify the integral but do not evaluate it. Sketch the region E.

6. (16pts) Use cylindrical coordinates to set up $\iiint_E \dfrac{z}{x+y+5}\,dV$, where E is the the region above the cone $z=\sqrt{2x^2+2y^2}$ and inside the sphere $x^2+y^2+z^2=6$. Do not evaluate the integral. Sketch the region E.

Bonus (10pts) Let $0 \leq a < b$, $u_1(x)$ and $u_2(x)$ be functions so that $u_1(x) \leq u_2(x)$ for all x in $[a, b]$, and let D be the region in the xz-plane between the graphs of $z = u_1(x)$ and $z = u_2(x)$, $a \leq x \leq b$. If we set $h(x) = u_2(x) - u_1(x)$, use cylindrical coordinates to show that the volume of the solid obtained by rotating the region D around the z-axis is

$$V = \int_a^b 2\pi x h(x)\, dx,$$

thereby verifying the formula for the shell method from Calculus 2.

Calculus 3 — Exam 4
MAT 309, Spring 2022 — D. Ivanšić

Name:______________________________

Show all your work!

1. (18pts) Let $\mathbf{F}(x,y)=\langle -y, y^2-1\rangle$.
a) Sketch the vector field by evaluating it at 9 points (for example, a 3×3 grid).
b) Is F conservative?
c) If C is the boundary of the square $[-1,1]\times[-1,1]$, traversed counterclockwise, is $\int_C \mathbf{F}\cdot d\mathbf{r}$ positive, negative, or zero? Explain your answer, then use the answer to give another justification to b).

2. (12pts) Let $\mathbf{F}(x,y)=\langle -2x, 1\rangle$. It is easy to see that $\mathbf{F}=\nabla f$, where $f(x,y)=y-x^2$. Apply the fundamental theorem for line integrals to:
a) Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, if C is the circle of radius 1, centered at $(1,1)$.
b) Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, if C is a curve from $(1,1)$ to $(3,4)$. Why is the curve not specified?

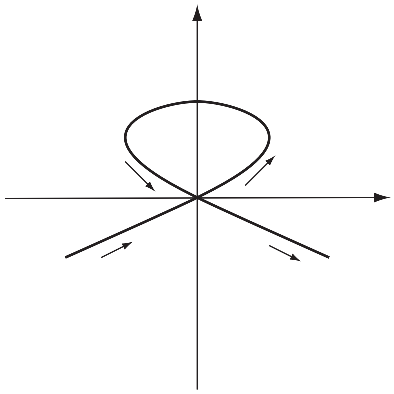
3. (26pts) In both cases set up and simplify the set-up, but do NOT evaluate the integral.
a) $\int_C (x^2 + y^2 + z^2)\, ds$, where C is the spiral $x = t\cos t$, $y = \sqrt{t}$, $z = t\sin t$, $0 \leq t \leq 4\pi$..
b) $\int_C \mathbf{F} \cdot d\mathbf{r}$, if $\mathbf{F}(x, y) = \langle xy, x - y \rangle$, where C is the line segment from $(-1, 0)$ to $(1, 1)$.

4. (10pts) Let $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$.
a) Compute $\mathbf{F} = \nabla f$.
b) Sketch the vector field $\mathbf{F}$. Little computation is needed, but pay attention to vector lengths.

5. (18pts) Consider the region D inside the circle $x^2 + y^2 = 4$ and above the lines $y = \sqrt{3}x$ and $y = -\sqrt{3}x$.
a) Draw the region.
b) Use Green's theorem to find the line integral $\int_C (-y^3 + 2xy^2)\,dx + (x^3 + 2x^2y)\,dy$, where C is the boundary of the region D, traversed counterclockwise.

6. (16pts) Let $\mathbf{F}(x, y) = \langle 3x^2 + 2y^2, 4xy + 6y \rangle$.
a) Is $\mathbf{F}$ is conservative? Your justification should say something about the domain.
b) If the field is conservative, find its potential function.

Bonus. (10pts) Pictured is the curve parametrized by $x(t) = t^3 - 4t$, $y(t) = 4 - t^2$. Use Green's theorem to find the area of the loop.



Calculus 3 — Final Exam
MAT 309, Spring 2022 — D. Ivanšić

Name: ____________________________

Show all your work!

1. (12pts) Find the equation of the plane that contains the point $(1, 3, -2)$ and the line given by parametric equations: $x = 1 - t$, $y = 4 - 3t$, $z = -4 + t$.

2. (18pts) Consider the function $f(x, y) = \dfrac{x^2}{y}$ on domain $\{(x, y) \mid y > 0\}$.

a) Sketch the contour map for the function, drawing level curves for levels $k = \frac{1}{2}, 1, 2, 0$.
b) At point $(-2, 2)$, find the directional derivative of f in the direction of $\langle 1, 1 \rangle$. In what direction is the directional derivative the greatest? What is the directional derivative in that direction?
c) Let $\mathbf{F} = \nabla f$. Sketch the vector field $\mathbf{F}$. If you did a), no computation is needed.

Apply the fundamental theorem for line integrals to answer:
d) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is part of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$?
e) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is a curve going from any point on level curve $k = 2$ to any point on level curve $k = \frac{1}{2}$?

3. (12pts) Find the equation of the tangent plane to the surface $y^2 + \frac{x \ln z}{z} = x + yz$ at point $(0, 1, 1)$.

4. (16pts) Find and classify the local extremes for $f(x, y) = y^3 + 6xy + x^2 - 18y - 6x$.

5. (15pts) Let D be the region bounded by the curves $y = x^3$, $x = 2$ and $y = 0$.
a) Sketch the region D.
b) Set up $\displaystyle\iint_D \frac{1}{(x^4+1)^2}\, dA$ as iterated integrals in both orders of integration.
c) Evaluate the double integral using the order you find easier.

6. (18pts) Sketch the region E that is under the paraboloid $z = 4 - x^2 - y^2$, above the xy-plane and in front of plane $x = 1$ (so points of the region satisfy $x \geq 1$). Then write the two iterated triple integrals that stand for $\iiint_E f\, dV$ which end in $dy\, dz\, dx$ and $dx\, dz\, dy$.

7. (20pts) Use cylindrical or spherical coordinates to evaluate the integral $\iiint_E x^2 + y^2 \, dV$, if E is the region that is inside the sphere $x^2 + y^2 + z^2 = 9$ and between the planes $y = x$ and $y = -x$, the part that intersects the positive x-axis. Sketch the region E.

8. (10pts) Set up and simplify the set-up, but do NOT evaluate the integral: $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the right half of the circle $x^2 + y^2 = 9$, traversed in the upward direction. and $\mathbf{F}(x, y) = \langle x + y, x - y \rangle$.

9. (16pts) Consider the triangle D with vertices $(0, 2)$, $(4, 2)$ and $(2, 0)$.
a) Draw the region.
b) Use Green's theorem to find the line integral $\int_C (3x^2y - y^2)\, dx + (x^3 + 2x^2)\, dy$, where C is the boundary of the triangle D, traversed counterclockwise.

10. (13pts) The surface area of a cone of radius r and height h is given by $S = \pi r\sqrt{r^2 + h^2}$ (bottom disk not included). Starting with a cone with radius 3 meters and height 4 meters, use differentials to estimate by how much the surface area changes if the radius decreases by 0.2 meters and height increases 0.1 meters.

Bonus (10pts) Suppose pollen is distributed in the plane with concentration $C(x, y) = x^2 + 2y^2$. A bee moving in the plane always tries to go in direction of the greatest increase of pollen concentration. Show that it will move along the curve $y = kx^2$ for some k. That is, show that a parametrization $\mathbf{r}(t)$ for this curve satisfies that $\mathbf{r}'(t)$ is always parallel to $\nabla C(\mathbf{r}(t))$.