Calculus 3 — Final Exam
MAT 309, Spring 2021 — D. Ivanšić

Name: ____________________

Show all your work!

1. (12pts) Find the equation of the plane that contains the lines given by parametric equations: $x = -2 - t$, $y = 12 + 3t$, $z = 2 + 2t$ and $x = 7 - 6t$, $y = 1 + 2t$, $z = -3 - t$. (These lines intersect — or they wouldn't determine a plane — but the point of intersection is not needed, so don't look for it.)

2. (20pts) Consider the function $f(x, y) = \dfrac{y}{x}$ on domain $\{(x, y) \mid x > 0\}$.
a) Sketch the contour map for the function, drawing level curves for levels $k = 0, \frac{1}{2}, 1, 2, -1, -\frac{1}{2}$.
b) At point $(3, -2)$, find the directional derivative of f in the direction of $\langle -1, 1\rangle$. In what direction is the directional derivative the greatest? What is the directional derivative in that direction?
c) Let $\mathbf{F} = \nabla f$. Sketch the vector field $\mathbf{F}$.
Apply the fundamental theorem for line integrals to answer:
d) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is part of the unit circle from $(0, 1)$ to $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$?
e) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is a curve going from any point on level curve $k = 3$ to any point on level curve $k = -2$?

3. (12pts) Find the equation of the tangent plane to the surface $x + y + z = e^{xyz}$ at point $(2, 0, -1)$.

4. (14pts) Find and classify the local extremes for $f(x, y) = x^2y + 2xy^2 + 3y$.

5. (16pts) Let D be the region bounded by the curves $y = e^x$, $y = e$ and $x = 0$.
a) Sketch the region D.
b) Set up $\displaystyle\iint_D \frac{1}{y}\, dA$ as iterated integrals in both orders of integration.
c) Evaluate the double integral using the order you find easier.

6. (18pts) Sketch the region E in the first octant $(x, y, z \geq 0)$ that is inside the cylinder $y^2 + z^2 = 4$ and "behind" the plane $y = 3x$. Then write the two iterated triple integrals that stand for $\iiint_E f\, dV$ which end in $dz\, dy\, dx$ and $dy\, dz\, dx$.

7. (14pts) Use either cylindrical or spherical coordinates to find the volume of a spherical cap E, the region inside the sphere $x^2 + y^2 + z^2 = 8$ that is above the plane $z = 2$. Sketch the region E.

8. (10pts) Set up and simplify the set-up, but do NOT evaluate the integral: $\int_C \frac{x+y}{xy+1}\, ds$, where C is the part of the curve $y = x^3 - x$ from $(-1, 0)$ to $(1, 0)$.

9. (20pts) Consider the region inside the circle $x^2 + y^2 = 4$ and above the lines $y = \sqrt{3}x$ and $y = -\frac{1}{\sqrt{3}}x$.
a) Draw the region.
b) Use Green's theorem to find the line integral $\int_C y^3\,dx + x^3\,dy$, where C is the boundary of the region D, traversed counterclockwise.

10. (12pts) The range of a projectile fired at angle α with initial velocity v is given by $R = \dfrac{v^2 \sin(2\alpha)}{10}$ (R is in meters, v in meters per second, α in radians). Use differentials to estimate the change in range of a projectile fired at 70 m/s at angle $\frac{\pi}{3}$ if velocity is increased by 5 meters per second, and angle is decreased by 0.2 radians.

Bonus. (10pts) Pictured is a spring 2020 friend from calculus 2, the curve parametrized by $x(t) = t^3 - 12t$, $y(t) = -t^2 - 2t + 8$. Use Green's theorem to find the area of the loop.