Calculus 3 — Exam 4
MAT 309, Spring 2021 — D. Ivanšić

Name: ____________________

Show all your work!

1. (16pts) Let $\mathbf{F}(x,y) = \langle x-y, x+y \rangle$.
a) Sketch the vector field by evaluating it at 9 points (for example, a 3×3 grid).
b) Is F conservative? Now, can you justify it just by looking at the picture?

2. (20pts) In both cases set up and simplify the set-up, but do NOT evaluate the integral.
a) $\int_C \frac{xyz}{x^2+z^2}\, ds$, where C is the helix $x = 3t$, $y = \cos t$, $z = \sin t$, $0 \le t \le 2\pi$..
b) $\int_C \mathbf{F} \cdot d\mathbf{r}$, if $\mathbf{F}(x,y) = \left\langle \frac{x-y}{x+y+4}, \frac{x+y}{x+y+4} \right\rangle$, where C is part of the circle $x^2+y^2=9$ from point $(0,3)$ to point $(-3,0)$, going the short way.

3. (16pts) Let $\mathbf{F}(x, y) = \langle 2x, 8y \rangle$. It is easy to see that $\mathbf{F} = \nabla f$, where $f(x, y) = x^2 + 4y^2$. Apply the fundamental theorem for line integrals to:

a) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is the circle of radius 2, centered at $(1, 0)$.

b) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is a curve from $(0, 0)$ to $(1, 2)$. (Why is the curve not specified?)

c) Sketch the directions of the vector field $\mathbf{F}$ by exploiting the function f. Very little computation is needed here.

4. (18pts) Consider the region D inside the triangle with vertices $(0, 0)$, $(2, 0)$ and $(2, 1)$.

a) Draw the region.

b) Use Green's theorem to find the line integral $\int_C (y \cos x - xy \sin x)\, dx + (xy + x \cos x)\, dy$, where C is the boundary of the region D, traversed counterclockwise. (Scary-looking, but it's not!)

5. (10pts) Suppose a particle moves in the velocity field $\mathbf{v}(x, y) = \langle x^2 - y^2, xy \rangle$. If it is at point $(1, 3)$ at time $t = 2$, estimate its location at time $t = 2.1$.

6. (20pts) Let $\mathbf{F}(x, y) = \left\langle \dfrac{2x}{x^2 + y}, e^y + \dfrac{1}{x^2 + y} \right\rangle$.

a) Find the domain of f: it has two parts, and consider the part that contains $(0, 1)$.

b) Compute $\dfrac{\partial Q}{\partial x}$ and $\dfrac{\partial P}{\partial y}$.

c) Is $\mathbf{F}$ is conservative? Your justification should say something about the domain.

d) If the field is conservative, find its potential function.

Bonus. (10pts) Pictured is a spring 2020 friend from calculus 2, the curve parametrized by $x(t) = t^3 - 12t$, $y(t) = -t^2 - 2t + 8$. Use Green's theorem to find the area of the loop.