Calculus 3 — Final Exam
MAT 309, Spring 2022 — D. Ivanšić

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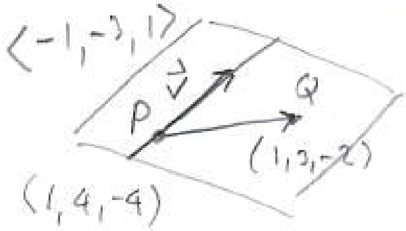
Show all your work!

1. (12pts) Find the equation of the plane that contains the point $(1,3,-2)$ and the line given by parametric equations: $x = 1-t$, $y = 4-3t$, $z = -4+t$.



Plane contains points $P=(1,4,-4)$ and $Q=(1,3,-2)$

$\vec{PQ} = \langle 0,-1,2\rangle$, $\vec{n} = \vec{v}\times\vec{PQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -3 & 1 \\ 0 & -1 & 2 \end{vmatrix}$

$= (-6+1)\vec{i} - (-2)\vec{j} + \vec{k}$

$= -5\vec{i} + 2\vec{j} + \vec{k}$

Take $\vec{n} = 5\vec{i} - 2\vec{j} - \vec{k}$

Eq. of plane:

$5(x-1) - 2(y-4) - (z-(-4)) = 0$

$\boxed{5x - 2y - z = 1}$

$-5+8-4 = -1$

2. (18pts) Consider the function $f(x,y) = \dfrac{x^2}{y}$ on domain $\{(x,y) \mid y>0\}$.

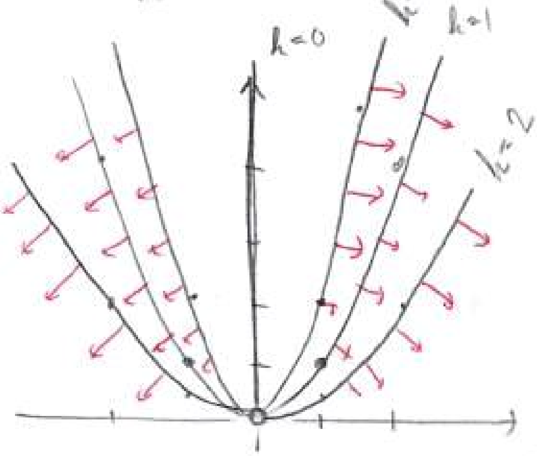
a) Sketch the contour map for the function, drawing level curves for levels $k = \frac{1}{2}, 1, 2, 0$.
b) At point $(-2,2)$, find the directional derivative of f in the direction of $\langle 1,1\rangle$. In what direction is the directional derivative the greatest? What is the directional derivative in that direction?
c) Let $\mathbf{F} = \nabla f$. Sketch the vector field $\mathbf{F}$. If you did a), no computation is needed.

Apply the fundamental theorem for line integrals to answer:
d) What is $\int_C \mathbf{F}\cdot d\mathbf{r}$, if C is part of the parabola $y = x^2$ from $(1,1)$ to $(2,4)$?
e) What is $\int_C \mathbf{F}\cdot d\mathbf{r}$, if C is a curve going from any point on level curve $k=2$ to any point on level curve $k=\frac{1}{2}$?

a) $\dfrac{x^2}{y} = k$ ($k \geq 0$ since $x^2 \geq 0$, $y > 0$)

c)

$y = \dfrac{x^2}{k}$ parabola for $k \neq 0$



$\dfrac{x^2}{y} = 0$

$x^2 = 0$, $x = 0$, positive y-axis for $k=0$

c) ∇f is perp. to level curves, in dir. of increasing k

b) $\nabla f = \left\langle \dfrac{2x}{y}, -\dfrac{x^2}{y^2}\right\rangle$

$\nabla f(-2,2) = \langle -2,-1\rangle$

$D_{\vec{u}} f = \langle -2,-1\rangle \cdot \underbrace{\dfrac{1}{\sqrt{1^2+1^2}}\langle 1,1\rangle}_{\vec{u}} = -\dfrac{3}{\sqrt{2}}$ $(-2-1)$

Greatest $D_u f$ in direction of ∇f, so $\langle -2,-1\rangle$, $D_u f = \sqrt{(-2)^2+(-1)^2} = \sqrt{5}$

d) $\int_C \nabla f \cdot d\vec{r} = f(2,4) - f(1,1) = 0$

both points on same level curve

e) $\int_C \nabla f\cdot d\vec{r} = f(B) - f(A)$

$= \dfrac{1}{2} - 2 = -\dfrac{3}{2}$

pts are where $f(x) = \frac{1}{2}$ and $f(x) = 2$

3. (12pts) Find the equation of the tangent plane to the surface $y^2 + \frac{x \ln z}{z} = x + yz$ at point $(0, 1, 1)$.

$F(x,y,z) = y^2 + \frac{x \ln z}{z} - x - yz$

$\vec{n} = \nabla F = \left\langle \frac{\ln z}{z} - 1,\ 2y - z,\ x \cdot \frac{\frac{1}{z} \cdot z - \ln z \cdot 1}{z^2} - y \right\rangle$

$= \left\langle \frac{\ln z}{z} - 1,\ 2y - z,\ \frac{x}{z^2}(1 - \ln z) - y \right\rangle$

take $\vec{n} =$

$\nabla F(0,1,1) = \left\langle \frac{0}{1} - 1,\ 2 - 1,\ \frac{0}{1^2}(1 - 0) - 1 \right\rangle = \langle -1, 1, -1 \rangle,$ $\quad \langle 1, -1, 1 \rangle$

$1 \cdot (x - 0) - 1(y - 1) + 1(z - 1) = 0$

$\boxed{x - y + z = 0}$

4. (16pts) Find and classify the local extremes for $f(x, y) = y^3 + 6xy + x^2 - 18y - 6x$.

$f_x = 6y + 2x - 6$

$f_y = 3y^2 + 6x - 18$

$\begin{cases} 6y + 2x - 6 = 0 \\ 3y^2 + 6x - 18 = 0 \end{cases}$ $\quad \begin{cases} 3y + x - 3 = 0 \\ y^2 + 2x - 6 = 0 \end{cases}$ $\Rightarrow$ $x = 3 - 3y$, put in 2nd eq.

$y^2 + 2(3 - 3y) - 6 = 0$

$y^2 - 6y = 0 \qquad y(y - 6) = 0$

$y = 0, 6$

$x = 3, -15$

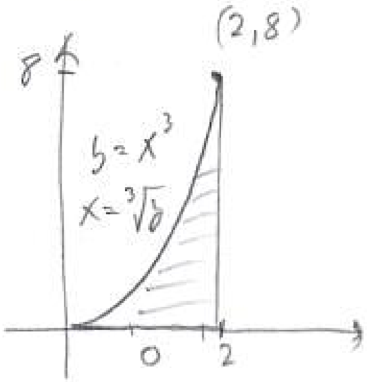
Candidates: $(3,0)$, $(-15,6)$

$D = \begin{vmatrix} 2 & 6 \\ 6 & 6y \end{vmatrix}$

$D(3,0) = \begin{vmatrix} 2 & 6 \\ 6 & 0 \end{vmatrix} = -36$ saddle point at $(3,0)$

$D(-15,6) = \begin{vmatrix} \boxed{2}^{>0} & 6 \\ 6 & 36 \end{vmatrix} = 72 - 36 = 36$ local min at $(-15,6)$

5. (15pts) Let D be the region bounded by the curves $y = x^3$, $x = 2$ and $y = 0$.
a) Sketch the region D.
b) Set up $\iint_D \frac{1}{(x^4+1)^2}\,dA$ as iterated integrals in both orders of integration.
c) Evaluate the double integral using the order you find easier.



Type 1: $\displaystyle\int_0^2\int_0^{x^3} \frac{1}{(x^4+1)^2}\,dy\,dx$

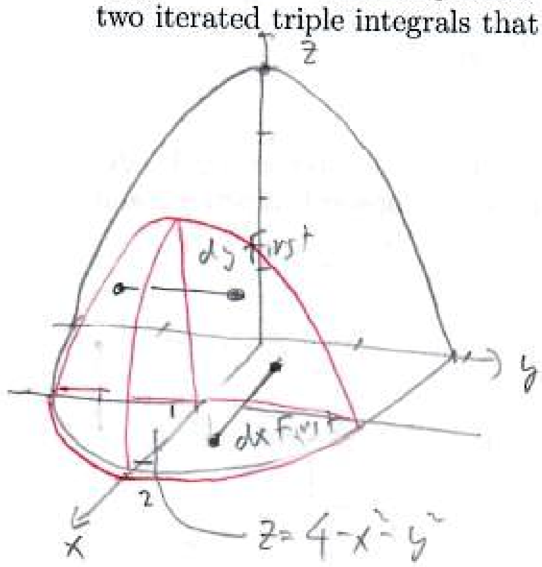
Type 2 $\displaystyle\int_0^8\int_{\sqrt[3]{y}}^{2} \frac{1}{(x^4+1)^2}\,dx\,dy$ ← hard to integrate by x

$$\int_0^2\int_0^{x^3} \frac{1}{(x^4+1)^2}\,dy\,dx = \int_0^2 \frac{1}{(x^4+1)^2}(x^3-0)\,dx$$

$$= \int_0^2 \frac{x^3}{(x^4+1)^2}\,dx = \begin{bmatrix} u = x^4+1 & x=2,\ u=17 \\ du = 4x^3dx & x=0,\ u=1 \\ \frac{du}{4} = x^3dx & \end{bmatrix} = \int_1^{17} \frac{1}{u^2}\frac{du}{4} = \frac{1}{4}\frac{u^{-1}}{-1}\Big|_1^{17} = -\frac{1}{4}\frac{1}{u}\Big|_1^{17}$$

$$= -\frac{1}{4}\left(\frac{1}{17}-\frac{1}{1}\right) = \frac{1}{4}\left(1-\frac{1}{17}\right) = \frac{1}{4}\cdot\frac{16}{17} = \frac{4}{17}$$

6. (18pts) Sketch the region E that is under the paraboloid $z = 4 - x^2 - y^2$, above the xy-plane and in front of plane $x = 1$ (so points of the region satisfy $x \geq 1$). Then write the two iterated triple integrals that stand for $\iiint_E f\,dV$ which end in $dy\,dz\,dx$ and $dx\,dz\,dy$.



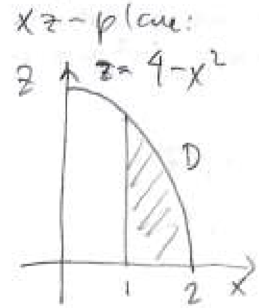
$z = 4-x^2-y^2$

$y = \pm\sqrt{4-x^2-z}$

$x = \pm\sqrt{4-y^2-z}$

Intersection of $\left.\begin{array}{l} z=4-x^2-y^2 \\ \text{and } x=1 \end{array}\right\}$ $z = 3-y^2$

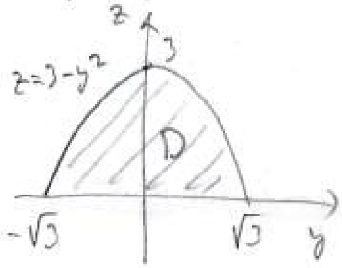
Projection to xz-plane:



$$\iiint_E f\,dV = \iint_D \int_{-\sqrt{4-x^2-z}}^{\sqrt{4-x^2-z}} f\,dy\,dA$$

$$= \int_1^2\int_0^{4-x^2}\int_{-\sqrt{4-x^2-z}}^{\sqrt{4-x^2-z}} f\,dy\,dz\,dx$$

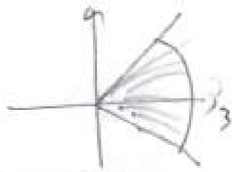
Projection to yz-plane



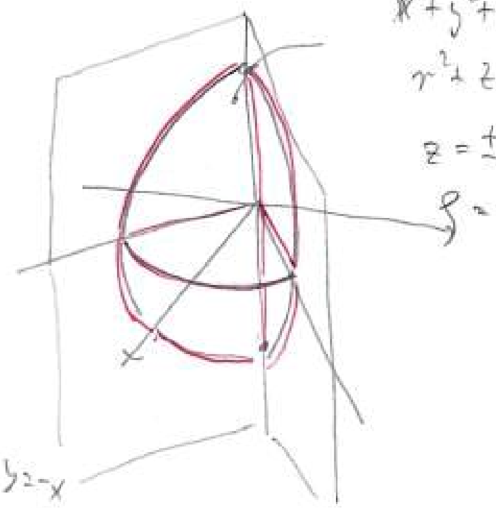
$$\iiint_E f\,dV = \iint_D \int_1^{\sqrt{4-y^2-z}} f\,dx\,dA$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}}\int_0^{3-y^2}\int_1^{\sqrt{4-y^2-z}} f\,dx\,dz\,dy$$

proj. to xy-plane



7. (20pts) Use cylindrical or spherical coordinates to evaluate the integral $\iiint_E x^2+y^2\,dV$, if E is the region that is inside the sphere $x^2+y^2+z^2=9$ and between the planes $y=x$ and $y=-x$, the part that intersects the positive x-axis. Sketch the region E.



$x^2+y^2+z^2=9$

$r^2+z^2=9$

$z=\pm\sqrt{9-r^2}$

$\rho=3$

$y=-x$, $\theta=-\frac{\pi}{4}$

$y=x$, $\theta=\frac{\pi}{4}$

Spherical $r=\rho\sin\phi$, $r^2=\rho^2\sin^2\phi$

$$\int_{-\pi/4}^{\pi/4}\int_0^{\pi}\int_0^3 r^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_{-\pi/4}^{\pi/4}\int_0^{\pi}\int_0^3 \rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta=\frac{\pi}{2}\cdot\int_0^{\pi}\sin^3\phi\,d\phi\int_0^3\rho^4\,d\rho$$

$\left[\begin{matrix}u=\cos\phi & \phi=\pi,\ u=-1\\ du=-\sin\phi\,d\phi & \phi=0,\ u=1\end{matrix}\right.$

$$=\frac{\pi}{2}\int_0^{\pi}(1-\cos^2\phi)\sin\phi\,d\phi\cdot\frac{\rho^5}{5}\Big|_0^3=\frac{\pi}{2}\cdot\frac{343}{5}\cdot\int_1^{-1}(1-u^2)(-du)$$

$$=\frac{343\pi}{10}\left(u-\frac{u^3}{3}\right)\Big|_{-1}^{1}=\frac{343\pi}{10}\cdot\left(1-(-1)-\frac{1}{3}(1-(-1))\right)=\frac{343\pi}{10}\cdot2\left(1-\frac{1}{3}\right)$$

$$=\frac{162\pi}{5}$$

Cylindrical:

$$\int_{-\pi/4}^{\pi/4}\int_0^3\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r^2\cdot r\,dz\,dr\,d\theta=\frac{\pi}{2}\int_0^3 r^3\left(\sqrt{9-r^2}-(-\sqrt{9-r^2})\right)dr$$

$r^2\,r$ $\quad$ $r^2=9-u$

$$=\pi\int_0^3 r^3\sqrt{9-r^2}\,dr=\left[\begin{matrix}u=9-r^2 & r=3,\ u=0\\ du=-2r\,dr & r=0,\ u=9\\ \frac{du}{-2}=r\,dr & \end{matrix}\right]=\pi\int_9^0(9-u)\sqrt{u}\,\frac{du}{-2}=\frac{\pi}{2}\int_0^9 9u^{\frac12}-u^{\frac32}\,du$$

$$=\frac{\pi}{2}\left(9\cdot\frac{2}{3}u^{\frac32}-\frac{2}{5}u^{\frac52}\right)\Big|_0^9$$

$$=\frac{\pi}{2}\left(6\left(9^{3/2}-0\right)-\frac{2}{5}\left(9^{\frac52}-0\right)\right)$$

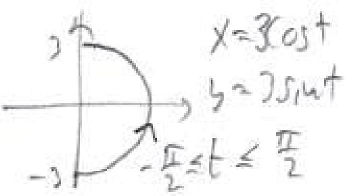
$9\cdot27$

$$=\frac{\pi}{2}\left(6\cdot27-\frac{2}{5}\cdot343\right)$$

$$=\frac{\pi}{2}\cdot\frac{30\cdot27-18\cdot27}{5}$$

$$=\frac{\pi}{\cancel{2}}\cdot\frac{\overset{6}{\cancel{12}}\cdot27}{5}=\frac{162\pi}{5}$$

8. (10pts) Set up and simplify the set-up, but do NOT evaluate the integral: $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the right half of the circle $x^2+y^2=9$, traversed in the upward direction. and $\mathbf{F}(x,y)=\langle x+y, x-y\rangle$.



$x=3\cos t$ $\quad x'=-3\sin t$

$y=3\sin t$ $\quad y'=3\cos t$

$-\frac{\pi}{2}\le t\le\frac{\pi}{2}$

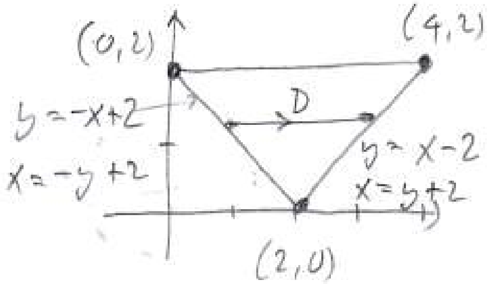
$$\int\vec{F}\cdot d\vec{r}=\int_{-\pi/2}^{\pi/2}\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$=\int_{-\pi/2}^{\pi/2}\langle 3\cos t+3\sin t,\ 3\cos t-3\sin t\rangle\cdot\langle-3\sin t,\ 3\cos t\rangle\,dt$$

$$=\int_{-\pi/2}^{\pi/2}-9\cancel{\cos t}\sin t-9\sin^2t+9\cos^2t-9\sin t\cos t=9\int_{-\pi/2}^{\pi/2}\cos^2t-\sin^2t-2\sin t\cos t\,dt$$

no (does not cancel)

9. (16pts) Consider the triangle D with vertices $(0,2)$, $(4,2)$ and $(2,0)$.
a) Draw the region.
b) Use Green's theorem to find the line integral $\int_C (3x^2y - y^2)\,dx + (x^3 + 2x^2)\,dy$, where C is the boundary of the triangle D, traversed counterclockwise.



P Q

$$\int_C (3x^2y - y^2)\,dx + (x^3+2x^2)\,dy$$

$$= \iint_D \frac{\partial}{\partial x}(x^3+2x^2) - \frac{\partial}{\partial y}(3x^2y - y^2)\,dA$$

$$= \iint_D (\cancel{3x^2} + 4x) - (\cancel{3x^2} - 2y)\,dA = \iint_D 4x + 2y\,dA$$

as type 2 region

$$= \int_0^2 \int_{-y+2}^{y+2} 4x+2y\,dx\,dy = \int_0^2 2x^2\Big|_{-y+2}^{y+2} + 2y\big((y+2)-(-y+2)\big)dy$$

$$= \int_0^2 2\big((y+2)^2 - (-y+2)\big)^2 + 2y\cdot 2y\,dy = \int_0^2 2\big(\cancel{y^2}+4y+\cancel{4} - (\cancel{4}-4y+\cancel{y^2})\big) + 4y^2\,dy$$

$$= \int_0^2 4y^2 + 16y\,dy = \left(\frac{4}{3}y^3 + 8y^2\right)\Big|_0^2 = \frac{4}{3}(2^3-0) + 8(2^2-0) = \frac{32}{3} + 32 = \frac{128}{3}$$

10. (13pts) The surface area of a cone of radius r and height h is given by $S = \pi r\sqrt{r^2+h^2}$ (bottom disk not included). Starting with a cone with radius 3 meters and height 4 meters, use differentials to estimate by how much the surface area changes if the radius decreases by 0.2 meters and height increases 0.1 meters.

$$dS = \frac{\partial S}{\partial r}\,dr + \frac{\partial S}{\partial h}\,dh$$

$$= \pi\left(\sqrt{r^2+h^2} + r\cdot\frac{2r}{2\sqrt{r^2+h^2}}\right)dr + \pi r\,\frac{2h}{2\sqrt{r^2+h^2}}\,dh$$

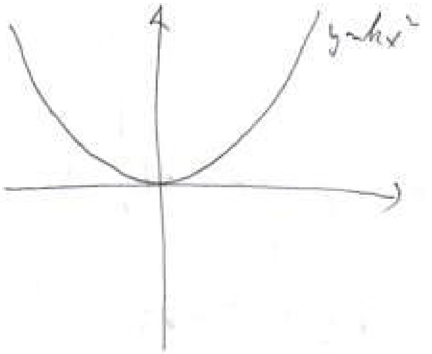
$$= \pi\left(\sqrt{r^2+h^2} + \frac{r^2}{\sqrt{r^2+h^2}}\right)dr + \pi\,\frac{rh}{\sqrt{r^2+h^2}}\,dh$$

$r=3$ $dr=-0.2$
$h=4$ $dh=0.1$
$\sqrt{3^2+4^2} = \sqrt{25} = 5$

$$dS = \pi\left(5+\frac{9}{5}\right)\cdot(-0.2) + \pi\cdot\frac{12}{5}\,0.1$$

$$= \pi\left(\frac{34}{5}\cdot\left(-\frac{1}{5}\right) + \frac{\cancel{12}^{\,6}}{5}\cdot\frac{1}{\cancel{10}_{\,5}}\right) = -\frac{28\pi}{25}\ m^2$$

Bonus (10pts) Suppose pollen is distributed in the plane with concentration $C(x, y) = x^2 + 2y^2$. A bee moving in the plane always tries to go in direction of the greatest increase of pollen concentration. Show that it will move along the curve $y = kx^2$ for some k. That is, show that a parametrization $\mathbf{r}(t)$ for this curve satisfies that $\mathbf{r}'(t)$ is always parallel to $\nabla C(\mathbf{r}(t))$.



$\nabla C = \langle 2x, 4y \rangle$

$\vec{r}(t) = \langle t, kt^2 \rangle$

$\vec{r}'(t) = \langle 1, 2kt \rangle$

$\nabla C(t, kt^2) = \langle 2t, 4kt^2 \rangle = 2t \cdot \langle 1, 2kt \rangle$

$= 2t \cdot \vec{r}'(t)$

Since $\nabla C = 2t\, \vec{r}'(t)$, ∇C and $\vec{r}'(t)$ are parallel