Calculus 3 — Exam 4
MAT 309, Spring 2022 — D. Ivanšić

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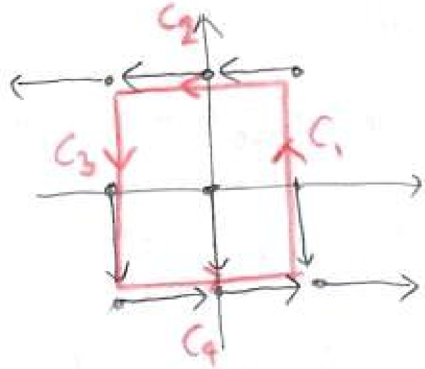
Show all your work!

1. (18pts) Let $\mathbf{F}(x, y) = \langle -y, y^2 - 1 \rangle$.
a) Sketch the vector field by evaluating it at 9 points (for example, a 3×3 grid).
b) Is F conservative?
c) If C is the boundary of the square $[-1, 1] \times [-1, 1]$, traversed counterclockwise, is $\int_C \mathbf{F} \cdot d\mathbf{r}$ positive, negative, or zero? Explain your answer, then use the answer to give another justification to b).

a)

(x,y)	$\vec{F}(x,y)$
$(-1,1)$	$\langle -1, 0 \rangle$
$(0,1)$	$\langle -1, 0 \rangle$
$(1,1)$	$\langle -1, 0 \rangle$
$(-1,0)$	$\langle 0, -1 \rangle$
$(0,0)$	$\langle 0, -1 \rangle$
$(1,0)$	$\langle 0, -1 \rangle$
$(-1,-1)$	$\langle 1, 0 \rangle$
$(0,-1)$	$\langle 1, 0 \rangle$
$(1,-1)$	$\langle 1, 0 \rangle$



b) $\dfrac{\partial Q}{\partial x} = 0 \quad \dfrac{\partial P}{\partial y} = -1$

Not equal, so cannot be conservative

c) It appears that $\int_C \vec{F} \cdot d\vec{r} \geq 0$

$\int_C = \int_{C_1} + \underbrace{\int_{C_2}}_{\geq 0} + \int_{C_3} + \underbrace{\int_{C_4}}_{\geq 0}$

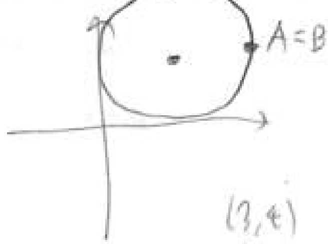
since $\vec{F} \cdot \vec{r} > 0$

$\int_{C_1}$ and $\int_{C_3}$ appear same but opposite, so cancel out, $\vec{F}$ field along lines $x = -1, +1$ same, traversed in opposite direct.

Since $\int_C \vec{F} \cdot d\vec{r} > 0$, on a closed path, $\vec{F}$ is not conservative

2. (12pts) Let $\mathbf{F}(x, y) = \langle -2x, 1 \rangle$. It is easy to see that $\mathbf{F} = \nabla f$, where $f(x, y) = y - x^2$. Apply the fundamental theorem for line integrals to:
a) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is the circle of radius 1, centered at $(1, 1)$.
b) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is a curve from $(1, 1)$ to $(3, 4)$. Why is the curve not specified?

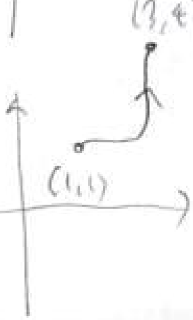
a)



$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A) = 0$

same point

b)



$\int_C \vec{F} \cdot d\vec{r} = f(3,4) - f(1,1) = (4 - 3^2) - (1 - 1^2) = -5$

Curve is not specified because integral does not depend on path.

3. (26pts) In both cases set up and simplify the set-up, but do NOT evaluate the integral.
a) $\int_C (x^2+y^2+z^2)\,ds$, where C is the spiral $x = t\cos t$, $y = \sqrt{t}$, $z = t\sin t$, $0 \le t \le 4\pi$..
b) $\int_C \mathbf{F}\cdot d\mathbf{r}$, if $\mathbf{F}(x,y) = \langle xy, x-y\rangle$, where C is the line segment from $(-1,0)$ to $(1,1)$.

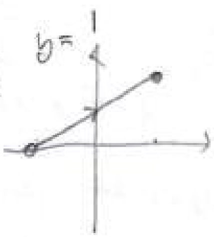
a) $\vec{r}\,'(t) = \langle \cos t - t\sin t, \frac{1}{2\sqrt{t}}, \sin t + t\cos t\rangle$

$|\vec{r}\,'(t)| = \sqrt{(\cos t - t\sin t)^2 + \left(\frac{1}{2\sqrt{t}}\right)^2 + (\sin t + t\cos t)^2}$

$= \sqrt{\cos^2 t - \cancel{2t\cos t\sin t} + t^2\sin^2 t + \frac{1}{4t} + \sin^2 t + \cancel{2t\sin t\cos t} + t^2\cos^2 t}$

$= \sqrt{1 + t^2(\sin^2 t + \cos^2 t) + \frac{1}{4t}} = \sqrt{1 + \frac{1}{4t} + t^2}$

$\int_C f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt = \int_0^{4\pi} (t^2\cos^2 t + (\sqrt{t})^2 + t^2\sin^2 t)\sqrt{1+\frac{1}{4t}+t^2} = \int_0^{4\pi} (t^2+t)\sqrt{1+\frac{1}{4t}+t^2}\,dt$

b) 

$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^1 \langle (2t-1)t, 2t-1-t\rangle\cdot\langle 2,1\rangle\,dt$

$\vec{r}(t) = (1-t)\langle -1,0\rangle + t\langle 1,1\rangle$

$= \int_0^1 \langle 2t^2-t, t-1\rangle\cdot\langle 2,1\rangle\,dt$

$= \langle t-1+t, t\rangle = \langle 2t-1, t\rangle \quad \vec{r}\,'(t) = \langle 2,1\rangle$

$= \int_0^1 4t^2-2t+t-1\,dt = \int_0^1 4t^2-t-1\,dt$

4. (10pts) Let $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$.
a) Compute $\mathbf{F} = \nabla f$.
b) Sketch the vector field $\mathbf{F}$. Little computation is needed, but pay attention to vector lengths.

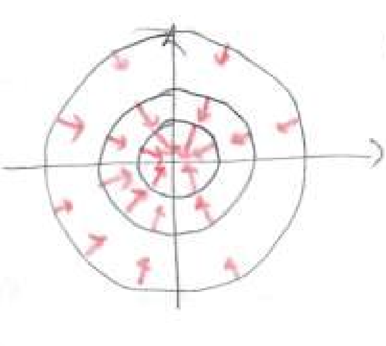
a) $f = (x^2+y^2)^{-\frac{1}{2}}$

$\nabla f = \langle -\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}}\cdot 2x, -\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}}\cdot 2y\rangle$

$= -\frac{1}{(x^2+y^2)^{3/2}}\langle x,y\rangle$

b) Perpendicular to level curves

$\frac{1}{\sqrt{x^2+y^2}} = k \qquad x^2+y^2 = \frac{1}{k^2}$



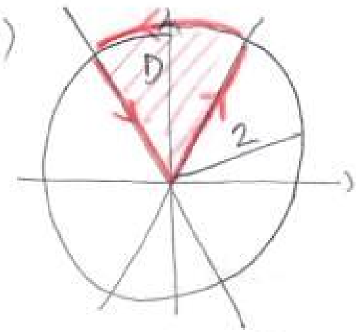
lengths get bigger toward origin

5. (18pts) Consider the region D inside the circle $x^2 + y^2 = 4$ and above the lines $y = \sqrt{3}x$ and $y = -\sqrt{3}x$.

a) Draw the region.

b) Use Green's theorem to find the line integral $\int_C (-y^3 + 2xy^2)\,dx + (x^3 + 2x^2y)\,dy$, where C is the boundary of the region D, traversed counterclockwise.

a)



$\tan\theta = \frac{y}{x} = \frac{\pm\sqrt{3}x}{x} = \pm\sqrt{3}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}$

b) $\int_C (-y^3+2xy^2)\,dx + (x^3+2x^2y)\,dy$

$= \iint_D \frac{\partial}{\partial x}(x^3+2x^2y) - \frac{\partial}{\partial y}(-y^3+2xy^2)\,dA$

$= \iint_D 3x^2+4xy-(-3y^2+4xy)\,dA$

$= \iint_D 3(x^2+y^2)\,dA = \left[\begin{matrix}\text{switch to}\\ \text{polar}\end{matrix}\right] = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\int_0^2 3r^2 r\,dr\,d\theta$

$= \frac{\pi}{3}\int_0^2 3r^3\,dr = \frac{\pi}{\cancel{3}}\cancel{3}\cdot\frac{r^4}{4}\Big|_0^2 = \frac{\pi}{4}\cdot(16-0) = 4\pi$

6. (16pts) Let $\mathbf{F}(x,y) = \langle 3x^2 + 2y^2, 4xy + 6y\rangle$.

a) Is $\mathbf{F}$ is conservative? Your justification should say something about the domain.

b ~~d~~) If the field is conservative, find its potential function.

a) $\left.\begin{matrix}\frac{\partial Q}{\partial x} = 4y\\ \frac{\partial P}{\partial y} = 4y\end{matrix}\right\}$ equal

Since domain is simply-connected (it is $\mathbb{R}^2$) Field is conservative

b) $f_x = 3x^2+2y^2$

$f = x^3+2y^2x+g(y)$

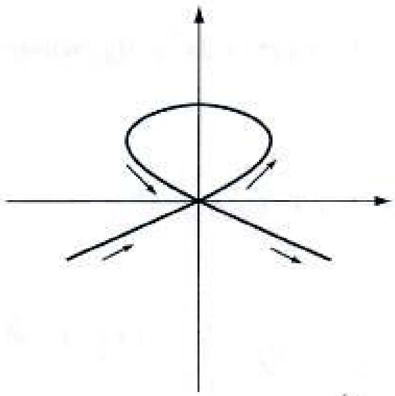
$\cancel{4xy}+6y = f_y = \cancel{4yx}+g'(y)$

$g'(y) = 6y$

$g(y) = 3y^2+C$

$f(x,y) = x^3+2xy^2+3y^2+C$

Bonus. (10pts) Pictured is the curve parametrized by $x(t) = t^3 - 4t$, $y(t) = 4 - t^2$. Use Green's theorem to find the area of the loop.



Self-intersection occurs when

$x = 0 = t^3 - 4t = t(t^2-4)$ $\quad t = 0, \pm 2$ $\quad$ satisfies both equations

$y = 0 = 4 - t^2$ $\quad t = \pm 2$

Loop is traced out for $-2 \le t \le 2$

$$\text{Area} = \iint_D 1\,dA = \int_C x\,dy = \int_{-2}^{2} (t^3-4t)\cdot(-2t)\,dt$$

$$= \int_{-2}^{2} -2t^4 + 8t^2\,dt = \left(-\frac{2}{5}t^5 + \frac{8}{3}t^3\right)\Big|_{-2}^{2}$$

$$= -\frac{2}{5}\left(2^5-(-2)^5\right) + \frac{8}{3}\left(2^3-(-2)^3\right) = -\frac{2}{5}\cdot 64 + \frac{8}{3}\cdot 16$$

$$= -\frac{128}{5} + \frac{128}{3} = \frac{-3\cdot 128 + 5\cdot 128}{15} = \frac{2\cdot 128}{15} = \frac{256}{15}$$