Calculus 3 — Exam 3
MAT 309, Spring 2022 — D. Ivanšić

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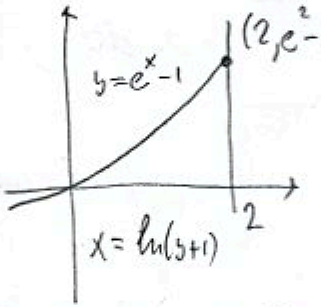
Show all your work!

1. (16pts) Let D be the region bounded by the curves $y = e^x - 1$, $y = 0$ and $x = 2$.
a) Sketch the region D.
b) Set up $\iint_D \sqrt{e^x - x}\, dA$ as iterated integrals in both orders of integration.
c) Evaluate the double integral using the easier order.



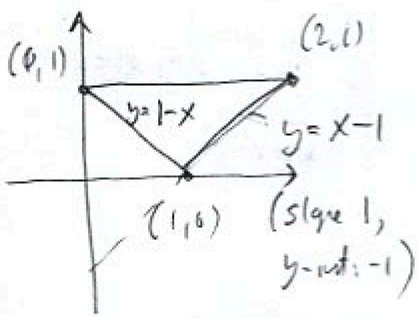
Type I: $\int_0^2 \int_0^{e^x-1} \sqrt{e^x-x}\, dy\, dx$

Type II $\int_0^{e^2-1} \int_{\ln(y+1)}^{2} \sqrt{e^x-x}\, dx\, dy$

$$\text{Type I} = \int_0^2 \sqrt{e^x-x}\,(e^x-1)\, dx = \left[\begin{matrix} u=e^x-x & x=2,\ u=e^2-2 \\ du=e^x-1 & x=0,\ u=1 \end{matrix}\right] = \int_1^{e^2-2} \sqrt{u}\, du$$

$$= \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{e^2-2} = \frac{2}{3}\left((e^2-2)^{\frac{3}{2}} - 1\right)$$

2. (12pts) Let D be the triangle with vertices $(1,0)$, $(0,1)$ and $(2,1)$. Set up $\iint_D \frac{x+y}{x^2+y^2}\, dA$, but do not evaluate the integral. Sketch the region of integration first.



Easier as type 2

$$\int_0^1 \int_{1-y}^{y+1} \frac{x+y}{x^2+y^2}\, dx\, dy$$

$y=1-x$ $\quad$ $y=x-1$
or $x=1-y$ $\quad$ or $x=y+1$

As type 1

$$\int_0^1 \int_{1-x}^{1} \frac{x+y}{x^2+y^2}\, dy\, dx + \int_1^2 \int_{x-1}^{1} \frac{x+y}{x^2+y^2}\, dy\, dx$$

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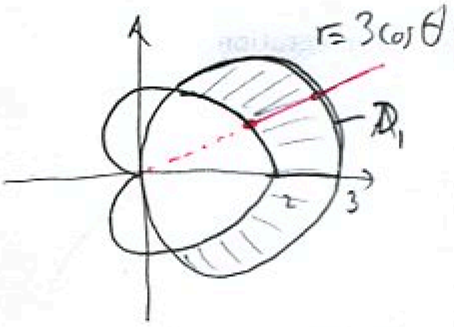
3. (22pts) Use polar coordinates to find the area of the region inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$. Sketch the region of integration first.



$$\text{Area} = 2\,\text{Area}(D_1) \quad (\text{by symmetry})$$

$$= 2\iint_{D_1} 1\, dA = 2\int_0^{\pi/3}\int_{1+\cos\theta}^{3\cos\theta} 1\cdot r\, dr\, d\theta$$

$$= 2\int_0^{\pi/3} \frac{r^2}{2}\Big|_{1+\cos\theta}^{3\cos\theta} d\theta = \int_0^{\pi/3} 9\cos^2\theta - (1+\cos\theta)^2\, d\theta$$

$$= \int_0^{\pi/3} 9\cos^2\theta - (1 + 2\cos\theta + \cos^2\theta)\, d\theta = \int_0^{\pi/3} 8\cos^2\theta - 2\cos\theta - 1\, d\theta$$

$$= \int_0^{\pi/3} 8\,\frac{1+\cos(2\theta)}{2} - 2\cos\theta - 1\, d\theta = \int_0^{\pi/3} 4\cos(2\theta) - 2\cos\theta + 3\, d\theta$$

$$= (2\sin 2\theta - 2\sin\theta)\Big|_0^{\pi/3} + 3\cdot\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} - 2\frac{\sqrt{3}}{2} + \pi = \pi$$

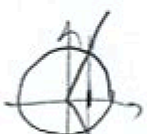
Intersection:

$3\cos\theta = 1 + \cos\theta$

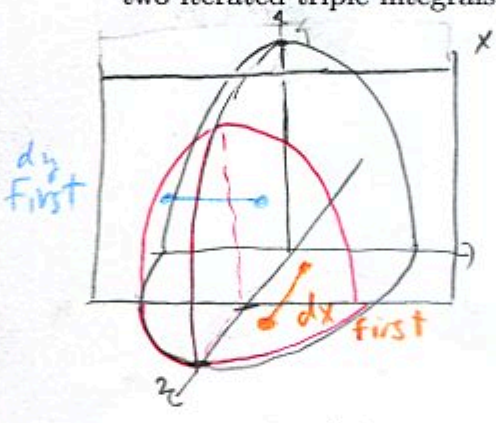
$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

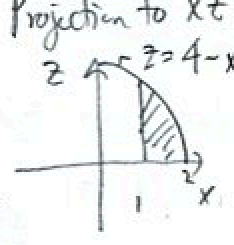
$\theta = \pm\frac{\pi}{3}$



4. (18pts) Sketch the region E that is under the paraboloid $z = 4 - x^2 - y^2$, above the xy-plane and in front of plane $x = 1$ (so points of the region satisfy $x \geq 1$). Then write the two iterated triple integrals that stand for $\iiint_E f\, dV$ which end in $dy\, dz\, dx$ and $dx\, dz\, dy$.



Projection to xz:



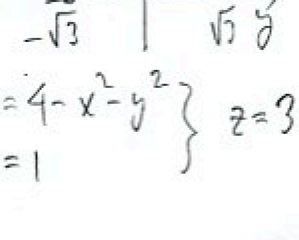
$$\int_1^2 \int_0^{4-x^2} \int_{-\sqrt{4-x^2-z}}^{\sqrt{4-x^2-z}} f\, dy\, dz\, dx$$

$z = 4 - x^2 - y^2$

$y = \pm\sqrt{4 - x^2 - z}$

$x = \pm\sqrt{4 - y^2 - z}$

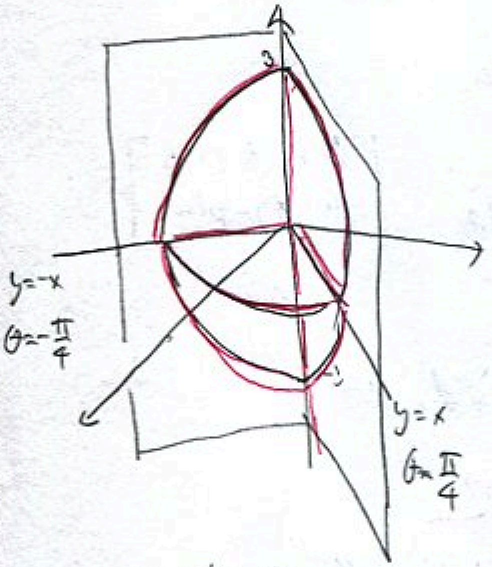
Projection to yz



$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_0^{3-y^2} \int_1^{\sqrt{4-y^2-z}} f\, dx\, dz\, dy$$

$$\left.\begin{array}{l} z = 4 - x^2 - y^2 \\ x = 1 \end{array}\right\} \; z = 3 - y^2$$

5. (16pts) Use spherical coordinates to set up the integral $\iiint_E x^2+y^2\,dV$, if E is the region that is inside the sphere $x^2+y^2+z^2=9$ and between the planes $y=x$ and $y=-x$, the part that intersects the positive x-axis. Simplify the integral but do not evaluate it. Sketch the region E.



$r = \rho\sin\phi$

$$\int_{-\pi/4}^{\pi/4}\int_0^{\pi}\int_0^{3} r^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

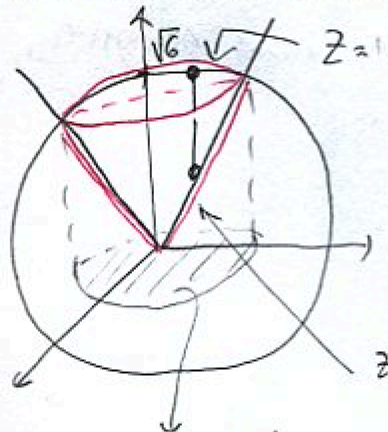
$$=\int_{-\pi/4}^{\pi/4}\int_0^{\pi}\int_0^{3} \rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta$$

$0\le\phi\le\pi$

$-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$

$0\le\rho\le 3$

6. (16pts) Use cylindrical coordinates to set up $\iiint_E \dfrac{z}{x+y+5}\,dV$, where E is the the region above the cone $z=\sqrt{2x^2+2y^2}$ and inside the sphere $x^2+y^2+z^2=6$. Do not evaluate the integral. Sketch the region E.

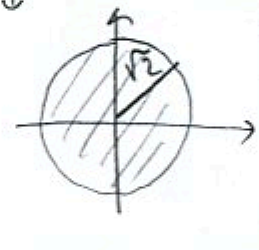


$z=\sqrt{6-x^2-y^2}=\sqrt{6-r^2}$

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{\sqrt{2}r}^{\sqrt{6-r^2}} \frac{z\,r}{r\cos\theta+r\sin\theta+5}\,dz\,dr\,d\theta$$

$z=\sqrt{2x^2+2y^2}=\sqrt{2}\sqrt{x^2+y^2}=\sqrt{2}\,r$

projection to xy-plane



$$\begin{cases} z=\sqrt{2x^2+2y^2} \\ x^2+y^2+z^2=6 \end{cases} \Rightarrow x^2+y^2+2x^2+2y^2=6$$

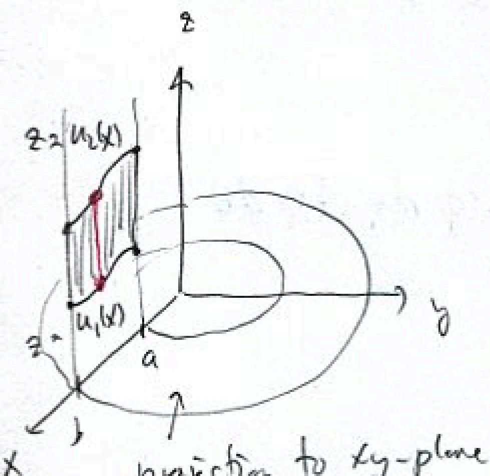
$3(x^2+y^2)=6$

$x^2+y^2=2$

Bonus (10pts) Let $0 \le a < b$, $u_1(x)$ and $u_2(x)$ be functions so that $u_1(x) \le u_2(x)$ for all x in $[a, b]$, and let D be the region in the xz-plane between the graphs of $z = u_1(x)$ and $z = u_2(x)$, $a \le x \le b$. If we set $h(x) = u_2(x) - u_1(x)$, use cylindrical coordinates to show that the volume of the solid obtained by rotating the region D around the z-axis is

$$V = \int_a^b 2\pi x h(x)\, dx,$$

thereby verifying the formula for the shell method from Calculus 2.



projection to xy-plane is a ring $a \le r \le b$

$$V = \iint_D \int_{u_1(r)}^{u_2(r)} 1 \cdot dz\, dA = \left[\begin{array}{l}D \text{ is a ring} \\ \text{in } xy\text{-plane}\end{array}\right]$$

$$= \int_0^{2\pi} \int_a^b \int_{u_1(r)}^{u_2(r)} 1 dz\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \underbrace{\int_a^b r \overbrace{(u_2(r) - u_1(r))}^{h(r)}\, dr}_{\text{const for } \theta}\, d\theta$$

$$= 2\pi \int_a^b r\, h(r)\, dr$$

$$= \int_a^b 2\pi r\, h(r)\, dr$$

which is the same as given,

Since the xz-plane is an rz-plane we can think of our region as