Calculus 3 — Exam 2
MAT 309, Spring 2022 — D. Ivanšić

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Show all your work!

1. (14pts) Let $f(x, y) = x^2 - y^2$.
a) Sketch the contour map for the function, drawing level curves for levels $k = -1, 0, 1$. Note the domain on the picture.
b) Without computation, draw the directions of $\nabla f(0, 1)$ and $\nabla f(1, 1)$. Note that these points are on the level curves you drew in a).

a) $x^2 - y^2 = -1$

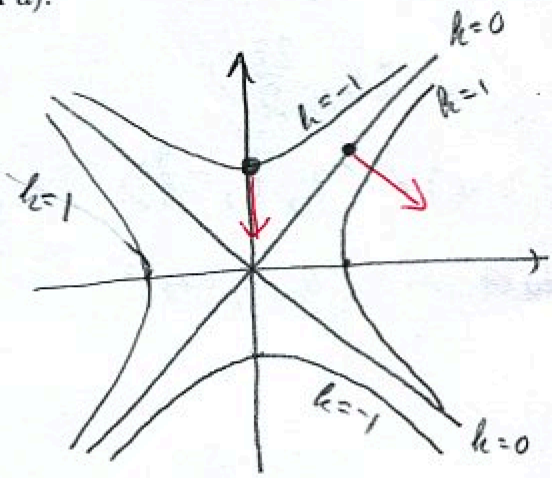
$y^2 - x^2 = 1$ hyperbola

$x^2 - y^2 = 0$

$y^2 = x^2$

$y = \pm x$ two lines

$x^2 - y^2 = 1$ hyperbola



b) ∇f is perpendicular to level curves, points in direction of greater k

2. (14pts) Mice are roaming around a flat board with a sticky surface, whose stickiness is given by the function $f(x, y) = (x^2 + y^2)e^{-3x}$. One mouse, located at point $(0, 1)$ sees another mouse located at point $(-1, 7)$ and starts moving towards it.
a) At first, does the mouse experience an increase or decrease in stickiness?
b) In what direction should the mouse at point $(0, 1)$ move in order to achieve the greatest stickiness decrease, and what is the rate of change of stickiness in that direction?

a) $\nabla f = \langle 2xe^{-3x} + (x^2+y^2)e^{-3x}(-3),\ 2ye^{-3x} \rangle = e^{-3x}\langle 2x - 3(x^2+y^2),\ 2y \rangle$

$\nabla f(0,1) = \underbrace{e^{-3\cdot 0}}_{=1}\langle 0 - 3(0+1),\ 2\cdot 1 \rangle = \langle -3, 2 \rangle$

direction from $(0,1)$ to $(-1,7)$ is $\langle -1-0,\ 7-1 \rangle = \langle -1, 6 \rangle$

$u = \frac{1}{\sqrt{(-1)^2+6^2}}\langle -1, 6 \rangle = \frac{1}{\sqrt{37}}\langle -1, 6 \rangle$. $D_u f = \langle -3, 2 \rangle \cdot \frac{1}{\sqrt{37}}\langle -1, 6 \rangle = \frac{1}{\sqrt{37}}(3+12)$

$= \frac{15}{\sqrt{37}} > 0$ so stickiness increasing

b) go in direction $-\nabla f = \langle 3, -2 \rangle$

rate of change is $-|\nabla f| = -\sqrt{3^2 + (-2)^2} = -\sqrt{13}$

3. (12pts) Find the equation of the tangent plane to the ellipsoid $\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{9}=10$ at the point $(1,2,-6)$. Simplify the plane equation to standard form.

$F(x,y,z) = \frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{9}$

$\nabla F = \langle 2x, \frac{1}{2}y, \frac{2z}{9}\rangle$

$\nabla F(1,2,-6) = \langle 2, 1, -\frac{4}{3}\rangle$

May take $\vec{n} = \langle 6, 3, -4\rangle$

Eq. of tan. plane:

$6(x-1)+3(y-2)-4(z-(-6))=0$

$6x+3y-4z=36$

4. (16pts) Let $z = \frac{\cos x}{\sin x+\cos y}$, $x = 2u-3v^2+6$, $y = u^2-4v-1$. Use the chain rule to find $\frac{\partial z}{\partial v}$ when $u=3$, $v=2$.

$$\frac{\partial z}{\partial x} = \frac{-\sin x(\sin x+\cos y)-\cos x\cdot\cos x}{(\sin x+\cos y)^2} = \frac{-\sin^2 x-\cos^2 x-\sin x(\cos y)}{(\sin x+\cos y)^2} = -\frac{1+\sin x(\cos y)}{(\sin x+\cos y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\cos x(\sin x+\cos y)^{-1} = \cos x(-1)(\sin x+\cos y)^{-2}(-\sin y) = \frac{\cos x\sin y}{(\sin x+\cos y)^2}$$

$\frac{\partial x}{\partial v} = -6v$

$\frac{\partial y}{\partial v} = -4$

$u=3,\ v=2$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$x=0,\ y=0$

when $u=3$, $v=2$

$x = 6-12+6=0$

$y = 9-8-1=0$

$$= -\frac{1+0\cdot1}{(0+1)^2}\cdot(-6\cdot2) + \frac{1\cdot0}{(0+1)^2}(-4)$$

$$= 12$$

5. (14pts) The surface area of a cone of radius r and height h is given by $S = \pi r\sqrt{r^2+h^2}$ (bottom disk not included). Starting with a cone with radius 6 meters and height 8 meters, use differentials to estimate by how much the surface area changes if the radius increases by 0.1 meters and height decreases 0.3 meters.

$$dS = \frac{\partial S}{\partial r}\,dr + \frac{\partial S}{\partial h}\,dh$$

$$= \pi\left(\sqrt{r^2+h^2} + r\cdot\frac{2r}{2\sqrt{r^2+h^2}}\right)dr + \pi r\frac{2h}{2\sqrt{r^2+h^2}}\,dh$$

$$= \pi\left(\sqrt{r^2+h^2} + \frac{r^2}{\sqrt{r^2+h^2}}\right)dr + \frac{\pi r h}{\sqrt{r^2+h^2}}$$

$$dS\Big|_{\substack{r=6 \;\; dr=0.1\\ h=8 \;\; dh=-0.3}} = \pi\left(\sqrt{6^2+8^2} + \frac{6^2}{\sqrt{6^2+8^2}}\right)\cdot 0.1 + \frac{\pi\cdot 6\cdot 8}{\sqrt{6^2+8^2}}(-0.3)$$

$$= \pi\left(10 + \frac{36}{10}\right)\cdot 0.1 - \pi\,\frac{48}{10}\,0.3$$

$$= \pi\cdot 13.6\cdot 0.1 - \pi\, 1.44 = \pi(1.36 - 1.44) = -0.08\pi$$

6. (10pts) Use implicit differentiation to find $\frac{\partial z}{\partial y}$ at the point $(e, 1, e)$, if
$\underbrace{x\ln y + y\ln z + z\ln x}_{F(x,y,z)} = e + 1.$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x\cdot\frac{1}{y} + \ln z}{y\cdot\frac{1}{z} + \ln x}$$

$$\frac{\partial z}{\partial y}\Big|_{(e,1,e)} = -\frac{\frac{e}{1}+1}{\frac{1}{e}+1} = -\frac{e+1}{\frac{1}{e}+1}\cdot\frac{e}{e} = -\frac{e(e+1)}{1+e} = -e$$

7. (20pts) Find and classify the local extremes for $f(x,y) = y^3 + 3x^2y - 3x^2 - \frac{15}{2}y^2$.

$f_x = 6xy - 6x$

$f_y = 3y^2 + 3x^2 - 15y$

$\begin{cases} 6xy - 6x = 0 \\ 3x^2 + 3y^2 - 15y = 0 \end{cases}$ $\begin{cases} x(y-1) = 0 \\ x^2 + y^2 - 5y = 0 \end{cases}$

First equation gives $x = 0$ or $y = 1$

$\underline{x = 0}$: $y^2 - 5y = 0$, $y(y-5) = 0$, $y = 0, 5$

$\underline{y = 1}$: $x^2 + 1 - 5 = 0$, $x^2 = 4$, $x = \pm 2$

$$D(x,y) = \begin{vmatrix} 6y-6 & 6x \\ 6x & 6y-15 \end{vmatrix}$$

(x,y)	$D(x,y)$	
$(0,0)$	$\begin{vmatrix} -6 & 0 \\ 0 & -15 \end{vmatrix} > 0$, $-6 < 0$	local max
$(0,5)$	$\begin{vmatrix} 24 & 0 \\ 0 & 15 \end{vmatrix} > 0$, $24 > 0$	local min
$(-2,1)$	$\begin{vmatrix} 0 & -12 \\ -12 & -9 \end{vmatrix} < 0$	saddle point,
$(2,1)$	$\begin{vmatrix} 0 & 12 \\ 12 & -9 \end{vmatrix} < 0$	saddle point,

Bonus (10pts) Suppose pollen is distributed in the plane with concentration $C(x,y) = x^2 + 2y^2$. A bee moving in the plane always tries to go in direction of the greatest increase of pollen concentration. Show that it will move along the curve $y = kx^2$ for some k. That is, show that a parametrization $\mathbf{r}(t)$ for this curve satisfies that $\mathbf{r}'(t)$ is always parallel to $\nabla C(\mathbf{r}(t))$.

A parametrization of $y = kx^2$ is

$x = t$

$y = kt^2$

$\nabla C(x,y) = \langle 2x, 4y \rangle$

$\vec{r}'(t) = \langle 1, 2kt \rangle$

$\nabla C(\vec{r}(t)) = \nabla C(t, kt^2) = \langle 2t, 4kt^2 \rangle = 2t \langle 1, 2kt \rangle = 2t \cdot r'(t)$

Therefore, $\nabla C(\vec{r}(t))$ and $\vec{r}'(t)$ are parallel.