Calculus 3 — Exam 1
MAT 309, Spring 2022 — D. Ivanšić

Name: Saul Ocean

Show all your work!

1. (11pts) Let $\mathbf{u} = \langle 2, -1, -3\rangle$ and $\mathbf{v} = \langle 4, 1, 0\rangle$.
a) Calculate $3\mathbf{u}$, $2\mathbf{u} - 5\mathbf{v}$, and $\mathbf{u}\cdot\mathbf{v}$.
b) Find a vector of length 4 in direction of $\mathbf{u}$.
c) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

a) $3\vec{u} = \langle 6, -3, -9\rangle$

$2\vec{u} - 5\vec{v} = \langle 4, -2, -6\rangle - \langle 20, 5, 0\rangle = \langle -16, -7, -6\rangle$

$\vec{u}\cdot\vec{v} = 8 - 1 + 0 = 7$

b) $|\vec{u}| = \sqrt{2^2 + (-1)^2 + (-3)^2} = \sqrt{4+1+9} = \sqrt{14}$

vector of length 4 in direction of $\vec{u}$ is

$4 \cdot \frac{1}{\sqrt{14}}\langle 2, -1, -3\rangle = \frac{4}{\sqrt{14}}\langle 2, -1, -3\rangle$

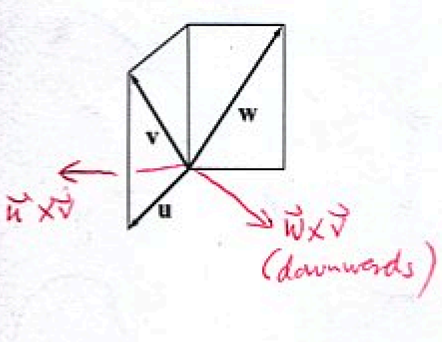
c) $\mathrm{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$

$= \frac{7}{4^2+1^2}\langle 4, 1, 0\rangle$

$= \frac{7}{17}\langle 4, 1, 0\rangle$

$= \langle \frac{28}{17}, \frac{7}{17}, 0\rangle$

2. (10pts) In the picture are two rectangles of width 2 units and height 3 units that are perpendicular to each other.
a) Draw the vectors $\mathbf{u}\times\mathbf{v}$ and $\mathbf{w}\times\mathbf{v}$, accurate length not being important.
b) What is the length of $\mathbf{u}\times\mathbf{w}$?



$|\vec{u}\times\vec{w}| = |\vec{u}||\vec{w}|\sin\frac{\pi}{2}$

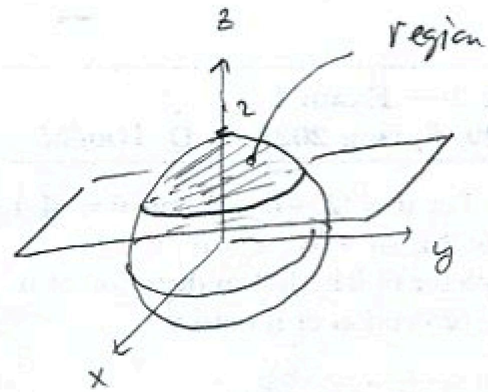
$= 2\cdot\sqrt{13}\cdot 1$

$= 2\sqrt{13}$

$|\vec{w}|^2 = 2^2 + 3^2$

$|\vec{w}| = \sqrt{13}$

region is spherical cap



3. (8pts) Draw the set in $\mathbf{R}^3$ described by:

$\underbrace{x^2+y^2+z^2<4}_{\text{inside of sphere centered at origin, radius 2}},\ \underbrace{z>1}_{\text{above plane } z=1}$

4. (13pts) Find the parametric equations of the line that is the intersection of planes $3x+y-z=2$ and $2x-y+4z=1$.

direction vector of line is $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -1 \\ 2 & -1 & 4 \end{vmatrix} = (4-1)\vec{i} - (12+2)\vec{j} + (-3-2)\vec{k}$
$= 3\vec{i} - 14\vec{j} - 5\vec{k}$

Point on plane: set $x=0$

$\begin{cases} y-z=2 \\ -y+4z=1 \end{cases}$

add: $3z=3$
$z=1$ $\quad y=2+z=3$

Point: $(0,3,1)$

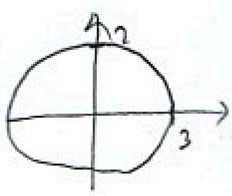
Eq. of line: $x = 3t$
$y = 3-14t$
$z = 1-5t$

5. (16pts) This problem is about the surface $-\dfrac{x^2}{9}+\dfrac{y^2}{9}+\dfrac{z^2}{4}=1$.
a) Identify and sketch the intersections of this surface with the coordinate planes.
b) Sketch the surface in 3D, with coordinate system visible.

a) yz plane, $x=0$

$\dfrac{y^2}{9}+\dfrac{z^2}{4}=1$

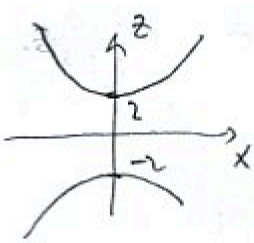
ellipse



b) xz-plane, $y=0$

$\dfrac{z^2}{4}-\dfrac{x^2}{9}=1$

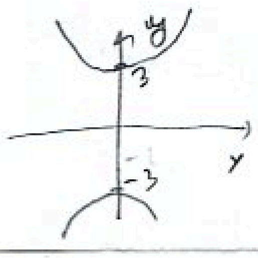
hyperbola

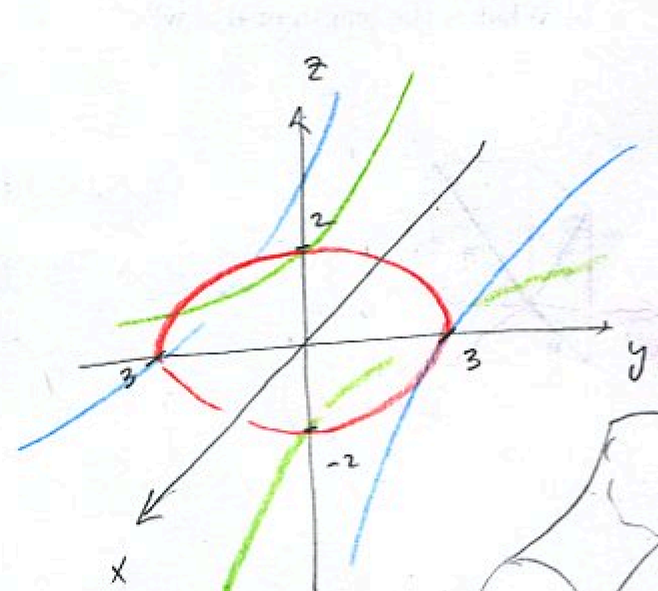


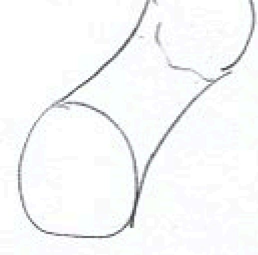
c) xy-plane, $z=0$

$\dfrac{y^2}{9}-\dfrac{x^2}{9}=1$

hyperbola



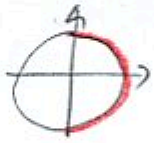




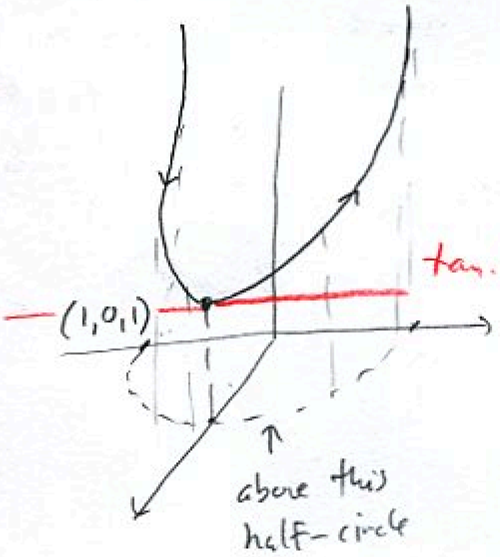
hyperboloid of one sheet

6. (14pts) The curve $\mathbf{r}(t) = \langle \cos t, \sin t, \sec t \rangle$ is given, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.
a) Sketch the curve in the coordinate system.
b) Find parametric equations of the tangent line to this curve when $t = 0$ and sketch the tangent line.

a) rotates in xy plane



Does $\cup$ in z coord



b) $\vec{r}\,'(t) = \langle -\sin t, \cos t, \sec t \tan t \rangle$

$\vec{r}(0) = \langle 1, 0, 1 \rangle$

$\vec{r}\,'(0) = \langle 0, 1, 0 \rangle$

$x = 1$
$y = t$
$z = 1$

7. (14pts) Find the length of the curve $\mathbf{r}(t) = \langle t^2 \sin t, t^2 \cos t, 2t \rangle$, $0 \le t \le 1$.

$\vec{r}\,'(t) = \langle 2t\sin t + t^2\cos t, \; 2t\cos t - t^2 \sin t, \; 2 \rangle$

$|\vec{r}\,'(t)| = \sqrt{(2t\sin t + t^2\cos t)^2 + (2t\cos t - t^2\sin t)^2 + 2^2}$

$= \sqrt{4t^2\sin^2 t + 4t^3\cancel{\sin t}\cos t + t^4\cos^2 t + 4t^2\cos^2 t - 4t^3\cancel{\sin t}\cos t + t^4\sin^2 t + 4}$

$= \sqrt{4t^2(\underbrace{\sin^2 t + \cos^2 t}_{=1}) + t^4(\underbrace{\cos^2 t + \sin^2 t}_{=1}) + 4} = \sqrt{4t^2 + t^4 + 4}$

$= \sqrt{(t^2+2)^2} = t^2 + 2$

$\int_0^1 |\vec{r}\,'(t)|\,dt = \int_0^1 t^2 + 2 \, dt = \left(\frac{t^3}{3} + 2t\right)\Big|_0^1 = \frac{1}{3} + 2 = \frac{7}{3}$

8. (14pts) A squishy is launched from point $(4, 1, 2)$ with initial velocity $\langle -2, 4, 7\rangle$.
a) Assuming gravity acts in the usual negative **k**-direction (let $g = 10$), find the vector function $\mathbf{r}(t)$ representing the position of the squishy.
b) When and at which point does the squishy hit the yz-plane?

a) $\vec{a} = \langle 0, 0, -10\rangle$

$\vec{v}(t) = \langle 0, 0, -10t\rangle + \vec{C}$

$\langle -2, 4, 7\rangle = \vec{v}(0) = 0 + \vec{C}$

$\vec{C} = \langle -2, 4, 7\rangle$

$\vec{v}(t) = \langle -2, 4, 7-10t\rangle$

$\vec{r}(t) = \langle -2t, 4t, 7t - 5t^2\rangle + \vec{D}$

$\langle 4, 1, 2\rangle = \vec{r}(0) = \vec{0} + \vec{D}$

$\vec{D} = \langle 4, 1, 2\rangle$

$\vec{r}(t) = \langle 4-2t, 1+4t, 2+7t-5t^2\rangle$

b) Strikes yz-plane when $x = 0$

$4 - 2t = 0$

$2t = 4$

$t = 2$

$2+14-20$

$\vec{r}(2) = \langle 0, 1+4\cdot 2, 2+7\cdot 2 - 5\cdot 2^2\rangle$

$= \langle 0, 9, -4\rangle$

Bonus (10pts) Find a unit vector that makes an angle of measure $\frac{\pi}{4}$ with **i** and an angle of measure $\frac{\pi}{3}$ with **k**. How many solutions are there?

Let $\vec{u} = \langle u_1, u_2, u_3\rangle$ be the desired vector.

We have $\cos\frac{\pi}{4} = \frac{\vec{u}\cdot\vec{i}}{|\vec{u}||\vec{i}|} = \frac{u_1\cdot 1 + u_2\cdot 0 + u_3 \cdot 0}{1\cdot 1}$

$\frac{\sqrt{2}}{2} = u_1$

$\cos\frac{\pi}{3} = \frac{\vec{u}\cdot\vec{k}}{|\vec{u}||\vec{k}|} = \frac{u_1\cdot 0 + u_2\cdot 0 + u_3\cdot 1}{1\cdot 1}$

$\frac{1}{2} = u_3$

Since $\vec{u}$ is unit, we have

$u_1^2 + u_2^2 + u_3^2 = 1$

$\left(\frac{\sqrt{2}}{2}\right)^2 + u_2^2 + \left(\frac{1}{2}\right)^2 = 1$

$\frac{2}{4} + u_2^2 + \frac{1}{4} = 1$

$u_2^2 = 1 - \frac{1}{4} - \frac{1}{2}$

$u_2^2 = \frac{1}{4}$

$u_2 = \pm\frac{1}{2}$

The two solutions are

$\vec{u} = \left\langle \frac{\sqrt{2}}{2}, \pm\frac{1}{2}, \frac{1}{2}\right\rangle$