13.1 Vector fields

Definition. Let D be a region in $\mathbf{R}^2$ or $\mathbf{R}^3$. A vector field is a function $\mathbf{F}$ that assigns to every point (x, y) or (x, y, z) in D a 2-dimensional vector $\mathbf{F}(x, y)$ or a 3-dimensional vector $\mathbf{F}(x, y, z)$. Often we write $\mathbf{F}(\mathbf{x})$ in vector notation.

Example. Sketch the 2-dimensional vector fields.

$\mathbf{F}(x, y) = x\,\mathbf{i} + y\,\mathbf{j}$, or $F(\mathbf{x}) = \mathbf{x}$

$\mathbf{F}_1(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\,\mathbf{j}$, or $F_1(\mathbf{x}) = \dfrac{\mathbf{x}}{|\mathbf{x}|}$

Example. Sketch the 2-dimensional vector fields.

$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

$F_1(x, y) = -\dfrac{y}{x^2 + y^2}\,\mathbf{i} + \dfrac{x}{x^2 + y^2}\,\mathbf{j}$

Example. The 3-dimensional gravitational vector field $\mathbf{G}(\mathbf{x}) = -\dfrac{GMm}{|\mathbf{x}|^3}\,\mathbf{x}$ gives the force of gravity from an object of mass M placed at the origin on an object of mass m at position $\mathbf{x}$. (G is the gravitational constant.)

Example. Let $\mathbf{a}$ be a constant vector in $\mathbf{R}^3$. We may define the vector field $\mathbf{F}(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$. Note that each vector $\mathbf{F}(\mathbf{x})$ is perpendicular to both $\mathbf{a}$ and $\mathbf{x}$. The vectors of the field are all in planes perpendicular to $\mathbf{a}$ and are the same in all those planes.

Example. Consider the velocity field of a gas flow, assuming it does not vary with time, for example flow of air around a wing flying at constant speed.

Example. If f is a scalar function then ∇f is a vector field. Draw the vector field ∇f if $f(\mathbf{x}) = |\mathbf{x}|^2$.

Note. The vector field ∇f is perpendicular to level curves (surfaces) of f.

Definition. If $\mathbf{F}$ is a vector field such that there exists a function f so that $\nabla f = \mathbf{F}$, we say that $\mathbf{F}$ is *conservative.* In this case, f is called the *potential function* of $\mathbf{F}$.

Example. The gravitational vector field $\mathbf{G}(\mathbf{x}) = -\dfrac{GMm}{|\mathbf{x}|^3}\mathbf{x}$ is conservative, its potential function is $g(\mathbf{x}) = \dfrac{GMm}{|\mathbf{x}|}$

13.2 Line Integrals

Example. Consider $f(x,y) = xy$ and the curve C, part of $y = x^2$ from $(0,0)$ to $(1,1)$. We wish to define the integral of f over C, $\int_C f(x,y)$.

Try different parametrizations of C and compute $\int_C f(x,y)$ in a naive way.

$x = t$, $y = t^2$, $0 \le t \le 1$:

$x = t^2$, $y = t^4$, $0 \le t \le 1$:

Problem: $\int_C f(x,y)$ computed in this way depends on the parametrization, and something called $\int_C f(x,y)$ ought to depend only on the curve C.

Definition. Let a curve C be parametrized by $x = x(t)$, $y = y(t)$, $a \le t \le b$. We define the *line integral of f over C* as:

$$\int_C f\,ds = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt$$

Example. Do the two parametrizations above give the same integral under this definition?

$x = t$, $y = t^2$, $0 \le t \le 1$:

$x = t^2$, $y = t^4$, $0 \le t \le 1$:

The same idea as in the example can be used to prove the general fact: the line integral $\int_C f\,ds$ does not depend on the parametrization, as long as C is traversed exactly once.

Note. $\int_C 1\, ds =$ length of C.

Note. $\displaystyle\int_C f\, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt$ in vector notation, which easily generalizes the definition to curves and functions in space.

Definition. If C is piecewise smooth, that is, it consists of smooth curves $C_1, C_2, \ldots, C_n$ that are in a chain, then

$$\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds + \cdots + \int_{C_n} f\, ds$$

Example. How much work is done by gravity as a roller-coaster wagon descends from A to B along curve C?

Note that only the component of the force in direction of motion does work.

The component of $\mathbf{F}$ in direction of the unit tangent vector $\mathbf{T}$ is $(\mathbf{F} \cdot \mathbf{T})\mathbf{T}$, so amount of force in direction $\mathbf{T}$ is $\mathbf{F} \cdot \mathbf{T}$. Since work = force × distance, to get total work done on curve C, we do $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, as the ds part includes the distance factor $|\mathbf{r}'(t)|$.

If the curve C is given by $\mathbf{r}(t)$, $a \leq t \leq b$, we compute:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds =$$

Definition. Let a curve C be parametrized by $\mathbf{r}(t)$, $a \leq t \leq b$, and let $\mathbf{F}$ be a vector field. We define the *line integral of* $\mathbf{F}$ *over* C as:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

which does not depend on the parametrization of C.

Interpretation: $\int_C \mathbf{F} \cdot d\mathbf{r}$ is work done by force field $\mathbf{F}$ acting on an object as it moves along curve C.

Example. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the helix $x = \cos t$, $y = \sin t$, $z = 3t$, $0 \leq t \leq 2\pi$ and $\mathbf{F}(x, y, z) = -y\,\mathbf{i} + x\,\mathbf{j} + \mathbf{k}$.

We know that the answer has to be positive, since the angle between force and tangent is less than $\frac{\pi}{2}$. If the object had traveled in the opposite direction, we would have gotten $W = \int_C \mathbf{F} \cdot d\mathbf{r} < 0$, since force acts against movement of object.

The changing of the sign of work when we reverse direction of travel along C illustrates the following general fact.

$$\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}$$

where $-C$ is the curve parametrized in the opposite direction. For an example where the parametrization of $-C$ is simple, take $\mathbf{r}(t)$, $0 \le t \le a$ as the parametrization of C. Then $-C$ is parametrized by $\mathbf{r}(a-t)$, $0 \le t \le a$.

Notation. Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, shorthand for $\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$. We then write

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$$

13.3 The Fundamental Theorem for Line Integrals

Theorem. Let $\mathbf{r}(t)$, $a \le t \le b$, be a smooth curve, $\mathbf{r}(a) = A$, $\mathbf{r}(b) = B$ and let f be differentiable, with ∇f continuous. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b) - f(\mathbf{r}(a)) = f(B) - f(A)$$

Proof.

Example. What work is done by the gravitational vector field $\mathbf{G}(\mathbf{x}) = -\frac{GMm}{|\mathbf{x}|^3}\mathbf{x}$ along a path from $(1,0,0)$ to $(1,1,3)$?

Independence of path

Suppose $\mathbf{F}$ is defined on some domain D and let C be any path in D joining points A and B. In general, $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends on C, so usually $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

If $\mathbf{F}$ is conservative, $\mathbf{F} = \nabla f$, so by the fundamental theorem for line integrals $\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$, that is, $\int_C \mathbf{F} \cdot d\mathbf{r}$ does not depend on the path C.

Definition. A *closed* path is a path with the same initial and terminal points.

Note. $\int_C \mathbf{F}\cdot d\mathbf{r}$ does not depend on path if and only if $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path C.

Proof.

Definition. A set is D in $\mathbf{R}^2$ or $\mathbf{R}^3$ is *open* if for every point P in D there is an open disk (ball) centered at P that is contained in D.

Examples.

Definition. A set is D in $\mathbf{R}^2$ or $\mathbf{R}^3$ is *connected* if every two points in D can be connected by a path inside D.

Examples.

Theorem. Let $\mathbf{F}$ be a vector field that is continuous on an open and connected region D. If $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path, then there exists a function f defined on D, such that $\nabla f = \mathbf{F}$.

Idea of proof.

Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a conservative field in the plane. Show that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

Example. Does the converse hold, that is, if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, is $\mathbf{F}$ conservative? Consider $F(x, y) = -\dfrac{y}{x^2+y^2}\,\mathbf{i} + \dfrac{x}{x^2+y^2}\,\mathbf{j}$.

a) Show that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. b) Show that $\int_C \mathbf{F}\cdot d\mathbf{r} = 2\pi$ on the unit circle C.

c) Conclude something.

The issue that keeps $\mathbf{F}$ from being conservative is that its domain is not simply-connected, that is, it has a "hole."

Definition. A curve is *simple* if it has no self-intersection.

Definition. A region D in $\mathbf{R}^2$ is *simply-connected* if
1) it is connected, and
2) every simple closed curve in D encloses points only in D.

Examples.

Theorem. Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field defined on an open, simply-connected planar region D, and suppose P and Q have continuous partial derivatives. If $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, then $\mathbf{F}$ is conservative.

Example. Are the following fields conservative? If so, find their potential function.

$\mathbf{F}(x, y) = y^2\,\mathbf{i} + x^2\,\mathbf{j}$

$\mathbf{F}(x, y) = (1 + 3x^2y)\,\mathbf{i} + (x^3 - y^2)\,\mathbf{j}$

13.4 Green's Theorem

A simple closed curve C in a plane divides the plane into two parts, bounded (D) and unbounded (outside of D). The positive orientation of C is the counterclockwise direction, or, more precisely: if we were to walk around the curve, the outside is on the right.

Green's Theorem. Let C be a positively oriented piecewise smooth simple closed curve in the plane, and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region containing D, then

$$\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

Note. We often use notation $\oint_C$ to indicate we are integrating over a simple closed curve in a positive direction.

Example. Compute $\oint_C x^3\,dx + xy^2\,dy$, where C consists of sides of the triangle with vertices $(0,0)$, $(1,1)$ and $(0,2)$.

Reading Green's theorem right to left, we see that we can transform a double integral of a particular form into a single integral.

Thus, if C is boundary of D, $\text{Area}(D) = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx.$

Example. Use the above to find the area of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Example. Green's theorem can be used to prove: if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on a simply-connected region D, then $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is conservative.

Example. Green's theorem can be used also for a region with holes. In this picture,

$$\oint_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

Example. Green's theorem can be used to simplify the curve of integration in $\int_C \mathbf{F} \cdot d\mathbf{r}$ when $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Let $F(x, y) = -\frac{y}{x^2 + y^2}\,\mathbf{i} + \frac{x}{x^2 + y^2}\,\mathbf{j}$, and let C be *any* simple closed curve that encloses the origin, oriented counterclockwise, Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$

What is $\int_C \mathbf{F} \cdot d\mathbf{r}$ for any simple closed curve that does not enclose the origin?

Note pattern of fundamental theorems in calculus:

$$\int_{\text{region}} \text{some kind of derivative of } F = \text{some kind of value of } F|_{\text{boundary of region}}$$

Observe this in:

Fundamental Theorem of Calculus

Fundamental Theorem of Line Integrals

Green's Theorem