Calculus 3 — Lecture notes
MAT 309, Spring 2022 — D. Ivanšić

12.1 Double integrals over rectangles

Recall the definition of the definite integral, inspired by trying to find the area under a curve:

$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$, if the limit exists, and it does for a wide class of functions (e.g. continuous).

Now let $f(x,y)$ be a function of two variables and consider a rectangle $R=[a,b]\times[c,d]$ in the plane. Subdivide $[a,b]$ and $[c,d]$ into m and n subintervals of equal length.

Form the double Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A, \text{ where } \Delta A = \Delta x \Delta y, \text{ area of the } ij\text{-th rectangle}$$

and consider what happens to the expression when $m,n\to\infty$. If the limit exists, it is called the *double integral* of f over the rectangle R. We define:

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

Interpretation of $\iint_R f(x,y)\,dA$ for $f(x,y) \geq 0$.

$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$ approximates the volume under the surface $z = f(x,y)$. Letting $m, n \to \infty$ improves the approximation, so

$$\iint_R f(x,y)\,dA = \text{ volume under the surface } z = f(x,y) \text{ and above the } xy\text{-plane}$$

If $f(x,y) < 0$ for some (x,y), the double integral counts volume above the xy-plane as positive and volume below the xy-plane as negative.

To see how to compute $\iint_R f(x,y)\,dA$, let $f(x,y) \geq 0$ on R. Then $\iint_R f(x,y)\,dA$ is the volume under the surface $z = f(x,y)$.

Fubini's Theorem. If $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

This is also true if f is bounded on R and discontinuous only on a finite number of curves and the iterated integrals exist.

Example. Find the volume under the paraboloid $z = x^2 + 3y^2$ above the rectangle $[-1, 3] \times [-2, 2]$.

Example. Compute $\iint_R y\cos(xy)\,dA$, where $R = [1,2] \times [0,\pi]$.

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12.2 Double Integrals over General Regions

Let f be defined on D, a bounded region in $\mathbf{R}^2$. We would like to define $\iint_D f(x,y)\,dA$.

We now define a function on some rectangle R that contains D:

$$F(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \text{ is in } D \\ 0, & \text{if } (x,y) \text{ is not in } D \end{cases} \quad \text{and set} \quad \iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

This makes sense because the value 0 does not contribute to the integral (consider volume under surface).

Typical regions of integration are:

Type 1 region of integration — between graphs of two functions of x.

Therefore, $\iint f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ for type 1 region.

Example. Evaluate $\iint_D x(y-1)\,dA$ if D is the region bounded by curves $y = x^2+1$, $y = 2x$ and $x = 0$.

Type 2 region of integration — between graphs of two functions of y.

$$\iint f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$ for type 2 region.

Example. Evaluate $\iint_D y\,dA$ if D is the region bounded by curves $x = y^2 + 2y$ and $y = x - 2$.

Example. Evaluate the integral $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$ by changing the order of integration.

Properties of double integrals

$$\iint_D f(x,y) + g(x,y)\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

$$\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$$

If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then

$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$$

If $D = D_1 \cup D_2$ and D_1 and D_2 do not overlap, except on their boundaries, then

$$\iint_{D_1} f(x,y) + \iint_{D_2} f(x,y)\,dA = \iint_D f(x,y)\,dA$$

$$\iint_D 1\, dA = \text{ area of the region } D$$

If $m \leq f(x,y) \leq M$ for all (x,y) in D, then

$$m \cdot \text{area}(D) \leq \iint_D f(x,y)\, dA \leq M \cdot \text{area}(D)$$

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12.3 Double Integrals in Polar Coordinates

Recall polar coordinates in the plane: $r =$ distance from the origin $\qquad x = r\cos\theta$
$\theta =$ angle with positive x-axis $\qquad y = r\sin\theta$

Consider integrals over the region D that is a "polar rectangle:" $a \leq r \leq b$ and $\alpha \leq \theta \leq \beta$.

If f is continuous on the polar rectangle D, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Reason:

Example. Compute the area of a disk of radius R.

The double integral over a more general polar region between rays $\theta = \alpha$, $\theta = \beta$ and polar curves $r = h_1(\theta)$ and $r = h_2(\theta)$ is given by

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

Recall some standard polar curves:

$$\left.\begin{array}{l} r = a\cos\theta \\ r = a\sin\theta \end{array}\right\} \text{ is a circle through origin, tangent to an axis, traversed once for } 0 \le \theta \le \pi$$

$$\left.\begin{array}{l} r = a\cos(n\theta) \\ r = a\sin(n\theta) \end{array}\right\} \text{ is a rose with } \left\{\begin{array}{l} n \text{ petals, if } n \text{ is odd (traversed once for } 0 \le \theta \le \pi) \\ 2n \text{ petals, if } n \text{ is even (traversed once for } 0 \le \theta \le 2\pi) \end{array}\right.$$

$$\left.\begin{array}{l} r = a(1 \pm \cos\theta) \\ r = a(1 \pm \sin\theta) \end{array}\right\} \text{ is a cardiod, traversed once for } 0 \le \theta \le 2\pi$$

Example. Find the volume under the cone $z = \sqrt{x^2 + y^2}$ above the rose $r = \sin(2\theta)$.

Example. Find the integral $\iint_D y \, dA$ if $D = \{(x, y) \mid x^2 + (y - 1)^2 \leq 1 \text{ and } x^2 + y^2 \geq 1\}$.

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12.5 Triple Integrals

The *triple integral* is defined much like the double integral. Suppose $f(x, y, z)$ is defined on a rectangular box $B = [a, b] \times [c, d] \times [r, s]$. Subdivide

$[a, b]$ into l pieces of width Δx
$[c, d]$ into m pieces of width Δy
$[r, s]$ into n pieces of width Δz

Choose point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in subbox B_{ijk}, form the triple Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V, \text{ where } \Delta V = \Delta x \Delta y \Delta z, \text{ volume of the } ijk\text{-th subbox}$$

and consider what happens to the expression when $l, m, n \to \infty$. If the limit exists, it is called the *triple integral* of f over the rectangular box B. We define:

$$\iiint_B f(x, y, z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V$$

Fubini's Theorem. If $f(x, y, z)$ is continuous on the rectangular box $B = [a, b]\times[c, d]\times[r, s]$, then

$$\iiint_B f(x, y, z)\,dV = \int_a^b \int_c^d \int_r^s f(x, y, z)\,dz\,dy\,dx = \iint_R \int_r^s f(x, y, z)\,dz\,dA$$

where $R = [a, b] \times [c, d]$. Five other orders of integrals are valid, too.

To integrate over a general bounded 3-dimensional region E, enclose it in a box B and define, like before

$$\iiint_E f(x, y, z)\,dV = \iiint_B F(x, y, z)\,dV$$

Note. $\iiint_E 1\,dV =$ volume of E.

Typical regions of integration are:

Type 1 region of integration — between graphs of two functions of x, y over region D in xy-plane

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dA$$

The inside-most integral by z becomes some function $G(x,y)$, whose integral $\iint_D G(x,y)\,dA$ is resolved in established ways.

Example. Compute $\iiint_E x\,dV$, if E is the region below the plane $2x+y-z=-1$, above the plane $z=-1$, and inside the cylinder bounded by surfaces $y=x^2$, $x=1$ and $y=0$.

Type 2 region of integration — between graphs of two functions of y, z over region D in yz-plane

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\,dA$$

Type 3 region of integration — between graphs of two functions of x, z over region D in xz-plane

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\,dA$$

Example. Find the volume of the solid inside both $y = x^2 + z^2 - 1$ and $x^2 + y^2 + z^2 = 7$.

Example. Express the integral $\iiint_E f(x, y, z)\, dV$ in six different ways, where E is bounded by the surfaces $z = 0$, $z = y$ and $x^2 = 1 - y$.

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12.6-7 Cylindrical and Spherical Coordinates

Cylindrical Coordinates

We obtain cylindrical coordinates (r, θ, z) by putting a polar coordinate system in the xy-plane and retaining z.

$$\begin{aligned} x &= r\cos\theta & \qquad r &= \sqrt{x^2+y^2} \\ y &= r\sin\theta & \qquad \tan\theta &= \frac{y}{x} \\ z &= z & \qquad z &= z \end{aligned}$$

Example. Convert coordinates.

$\left(1, \frac{3\pi}{4}, -3\right) \to$ Cartesian

$(-5, -3, 6) \to$ cylindrical

Example. Sketch the surfaces.

$r = r_0$

$\theta = \theta_0$

$z = z_0$

Example. Sketch the surface $r^2 + z^2 = 16$.

Spherical Coordinates

In spherical coordinates (ρ, θ, ϕ), $\rho \geq 0$ is distance from origin, θ has same meaning as in cylindrical coordinates and ϕ is angle from positive z-axis $(0 \leq \phi \leq \pi)$.

$$x = \rho \sin\phi \cos\theta \qquad \rho = \sqrt{x^2 + y^2 + z^2}$$
$$y = \rho \sin\phi \sin\theta \qquad \tan\theta = \frac{y}{x}$$
$$z = \rho \cos\phi \qquad \cos\phi = \frac{z}{\rho}$$

Note that
$r = \rho \sin\phi$

Example. Convert coordinates.

$\left(3, \frac{\pi}{4}, \frac{2\pi}{3}\right) \to$ Cartesian

$(-\sqrt{3}, 1, 2\sqrt{3}) \to$ spherical

Example. Sketch the surfaces.

$\rho = \rho_0$

$\theta = \theta_0$

$\phi = \phi_0$

Example. Sketch the surface $\rho = \sin\phi$.

Triple integrals in cylindrical coordinates. If E is the region between surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$ whose projection to the xy-plane is the polar region $\alpha \le \theta \le \beta$, $h_1(\theta) \le r \le h_2(\theta)$, then

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Example. Find the volume cut out of a ball of radius b by a cylinder of radius a ($a < b$) whose axis contains a diameter of the ball.

Triple integrals in spherical coordinates. If E is the spherical wedge $a \leq \rho \leq b$, $\alpha \leq \theta \leq \beta$, $c \leq \phi \leq d$, then

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

Example. Find $\iiint_E z^2 \, dV$ if E is the upper half of the ball $x^2 + y^2 + z^2 \leq 4$.

Example. Find the volume inside the cone $z^2 = 3x^2+3y^2$ and the sphere $x^2+y^2+(z-1)^2 = 1$.