Calculus 3 — Lecture notes
MAT 309, Spring 2022 — D. Ivanšić

11.1 Functions of Several Variables

Many quantities depend on more than one variable. Consider these quantities, which depend on two variables.

Example.

1) $f(x,y) = \sqrt{x^2+y^2}$, distance from a point with coordinates (x,y) to the origin.

2) $A(l,w) = lw$, area of a rectangle with length l and width w.

3) $h(v,\alpha) = \dfrac{v^2 \sin(2\alpha)}{g}$, range of a projectile launched at angle α and initial velocity v.

In general, a function of two variables $f(x,y)$ is a rule that assigns to each point (x,y) in a domain D a real number $f(x,y)$.

The *range* of the function f is the set $\{f(x,y) \mid (x,y) \in D\}$ (all numbers $f(x,y)$, as (x,y) goes through the domain D).

Find the domain and range of the following functions.

Example. $f(x,y) = \sqrt{x^2+y^2}$

Example. $f(x,y) = \ln(y - x^3)$

Definition. The *graph* of a function of two variables $f(x, y)$ is the set of all points (x, y, z) such that $z = f(x, y)$ and (x, y) is in the domain of the function.

Example. Sketch the graph of $f(x, y) = 2x + y - 1$.
(Note: a general linear function in two variables is $f(x, y) = ax + by + c$).

Example. Sketch the graph of $f(x, y) = x^2 + 3y^2$ using traces in planes $z = k$.

Definition. *Level curves* of a function of two variables $f(x, y)$ are curves whose equation is $f(x, y) = k$, for some real number k.

Note. Level curves are projections to the xy-plane of traces of $z = f(x, y)$ in $z = k$.

Example. Isothermals are level curves of the temperature function.

Example. Sketch the level curves of the function $g(x, y) = ye^x$.

For functions of 3 variables $f(x, y, z)$, we can also consider domain, range and level surfaces. Drawing graphs is not possible, because the graph would be in 4-dimensional space, but we can gain some insight into behavior of the function by studying its *level surfaces*, surfaces where $f(x, y, z) = k$.

Example. Find the domain, range and sketch level surfaces for the function $f(x, y, z) = \sqrt{x^2 - y^2 - z^2}$.

Functions of n variables are considered in a similar way.

Example. The average of n numbers is $f(x_1, x_2, \ldots, x_n) = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$.

Depending on context, we may think of a function $f(x_1, x_2, \ldots, x_n)$ as:

1) a function of n variables

2) a function of a point variable $(x_1, x_2, \ldots, x_n)$

3) a function of a vector variable $\langle x_1, x_2, \ldots, x_n \rangle$

Example. By extending the notion of dot product to vectors with n components, we can consider the function

$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$, where $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$, $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$.

Thus, $f(\mathbf{x}) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$, a linear function of n variables.

11.3 Partial Derivatives

Example. The volume of a cylinder of radius x and height y is $V(x,y) = \pi x^2 y$.

If we fix y, e.g. $y = 3$ we get a function only of x, $g(x) = V(x,3) =$.

We may find $\dfrac{dg}{dx} =$, rate of change of V with respect to x, when y is fixed at 3.

If we fix x, e.g. $x = 2$ we get a function only of y, $h(y) = V(2,y) =$.

We may find $\dfrac{dh}{dy} =$, rate of change of V with respect to y, when x is fixed at 2.

For general fixed x or y:

$g(x) = \pi x^2 y$ (y fixed), $\dfrac{dg}{dx} =$, partial derivative of V with respect to x

$h(y) = \pi x^2 y$ (x fixed), $\dfrac{dh}{dy} =$, partial derivative of V with respect to y

More precisely, we have

Definition. Suppose $f(x,y)$ is a function of two variables. We define:

$f_x(a,b) = \lim\limits_{h\to 0} \dfrac{f(a+h,b) - f(a,b)}{h}$, the partial derivative of f with respect to x at (a,b)

$f_y(a,b) = \lim\limits_{h\to 0} \dfrac{f(a,b+h) - f(a,b)}{h}$, the partial derivative of f with respect to x at (a,b)

In general, if $f(x_1, x_2, \ldots, x_n)$ is a function of n variables we define

$f_{x_i}(a_1, a_2, \ldots, a_n) = \lim\limits_{h\to 0} \dfrac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, a_2, \ldots, a_n)}{h}$,
the partial derivative of f with respect to x_i at $(a_1, a_2, \ldots, a_n)$

Notation.

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = D_x f$$

To find a partial derivative by a variable, treat all the other variables as constants and differentiate in the usual way.

Example. Find the partial derivatives of f by every variable.

$f(x, y) = x^2 - 3x^3y^3 + 4x \ln y$.

$f(x, y, z) = e^{xy} \cos(y^2 + z^2)$

Graphical interpretation of partial derivatives.

$f_x(a, b)$ is the slope of the curve that is the intersection of the plane $y = b$ with the graph of $f(x, y)$.

$f_y(a, b)$ is the slope of the curve that is the intersection of the plane $x = a$ with the graph of $f(x, y)$.

As we have seen from examples, f_x, f_y or f_z are functions of several variables in their own right, so we can take their partial derivatives. The notation for these second partial derivatives is: $f_{xx} = (f_x)_x$, $f_{xy} = (f_x)_y$, etc.

Example. Find the second partial derivatives of $f(x, y) = x^2 - 3x^3y^3 + 4x \ln y$ by every variable combination.

Example. For $f(x,y,z) = e^{xy}\cos(y^2+z^2)$, find the second partial derivatives f_{xx}, f_{xz}, f_{zx}, f_{xy} and f_{yx}. What do you notice?

Clairaut's theorem. Suppose f is defined on a disk D that contains (a,b). If f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$. That is, we can take higher partial derivatives of functions in arbitrary order.

11.4 Tangent Planes and Linear Approximations

It is intuitively clear that a smooth surface has a tangent plane. For example, placing a thin board so it touches a vase mimics the idea of the tangent plane.

Consider a curve on the surface. At any point of the curve, the tangent line of the curve ought to lie in the tangent plane of the surface at that point.

Now specialize this to:
– the surface $z = f(x, y)$
– the curves resulting from intersecting the surface with vertical planes $x = x_0$ and $y = y_0$.

The tangent lines to those curves at point (x_0, y_0) are in the tangent plane to the surface at (x_0, y_0) — we use this fact to to get a normal vector of the tangent plane.

The equation of the tangent plane to graph of function $z = f(x, y)$ at point (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example. Find the equation of the tangent plane to the sphere $z = \sqrt{16 - x^2 - y^2}$ at point $(2, -3, \sqrt{3})$.

Note. Suppose $z = L(x, y)$ is the equation of the tangent plane to the surface at point (x_0, y_0). Then for $(x, y) \approx (x_0, y_0)$ we have $L(x, y) \approx f(x, y)$.

$(x, y) \approx (x_0, y_0)$ means distance between points (x, y) and (x_0, y_0) is small.

Example. If $L(x, y)$ is the tangent plane of the above example, compute the following and compare:

$L(2.1, -2.9) =$

$f(2.1, -2.9) =$

Equation of the tangent plane can be rewritten:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$dz = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

We introduce the *differential* dz as a function of two variables dx and dy that represent the change in x- and y-values as we move away from a point (x_0, y_0). (Thus, $dx = \Delta x$ and $dy = \Delta y$.)

$$dz = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy$$

dz represents change in value of $L(x, y)$, the tangent plane at (x_0, y_0, z_0)
Δz represents $f(x_0 + dx, y_0 + dy) - (x_0, y_0)$, change in value of the function $f(x, y)$

Since $dz \approx \Delta z$, dz can be used to approximate the change in value Δz of the function $f(x, y)$.

Example. The radius $x = 30$cm and height $y = 50$cm of a cylinder were measured with accuracy 0.5cm and 0.2cm, respectively. Use differentials to estimate the maximal error in measuring the surface area of the cylinder $A = 2\pi x^2 + 2\pi xy$. Find the percentage error estimate as well.

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11.5 The Chain Rule

Recall: if $y = y(x)$ and $x = x(t)$, then y is a function of t via x and

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \qquad \left(= \frac{dy}{dx}(x(t)) \cdot \frac{dx}{dt}(t) \right) \qquad \text{Chain rule}$$

Suppose that $z = z(x, y)$, $x = x(t)$, $y = y(t)$. This makes z a function of t via x and y, $z(t) = z(x(t), y(t))$, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\left(\text{Really: } \frac{dz}{dt} = \frac{\partial z}{\partial x}(x(t), y(t)) \cdot \frac{dx}{dt}(t) + \frac{\partial z}{\partial y}(x(t), y(t)) \cdot \frac{dy}{dt}(t) \right)$$

Example. Let $z = \ln(x + y^2)$, $x = \sqrt{2+t}$, $y = t^3$. Find $\dfrac{dz}{dt}$ when $t = 2$.

Suppose that $z = z(x, y)$, $x = x(s, t)$, $y = y(s, t)$. This makes z a function of s and t via x and y, $z(s, t) = z(x(s, t), y(s, t))$, and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

evaluated at appropriate numbers.

Example. Let $z = x^2 \sin y$, $x = s^2 + t^2$, $y = 2st$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ when $s = \dfrac{\pi}{4}$, $t = 1$.

In general, let $u = u(x_1, \dots x_n)$ — u is a function of n variables
$x_i = x_i(t_1, \dots t_m)$ — each of $x_1, \dots, x_n$ is a function of m variables

This makes u a function of $t_1, \dots t_m$ via $x_1, \dots, x_n$, and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$$

$$\frac{\partial u}{\partial t_i} = \left\langle \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right\rangle \cdot \left\langle \frac{\partial x_1}{\partial t_i}, \frac{\partial x_2}{\partial t_i}, \dots, \frac{\partial x_n}{\partial t_i} \right\rangle$$

The dependence of variables may be visualized using the tree diagram at left.

Then $\dfrac{\partial u}{\partial t_i}$ is the sum of products of partial derivatives along every path from u to t_i.

Implicit differentiation

Example. If $x^2 + (y-1)^2 = 4$, then y is implicitly a function of x. It is in form $F(x, y) = k$, where $F(x, y) = x^2 + (y-1)^2$, $k = 4$.

Differentiate $F(x, y) = k$ by x:

If y is given implicitly as a function of x via the equation $F(x, y) = k$, then

$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Compute $\dfrac{dy}{dx}$ for our example and note where the formula is valid.

Example. If $x^2 + y^2 - z^2 = 1$, then z is implicitly a function of x and y. This equation is in form $F(x, y, z) = k$. Differentiate it by x and y:

If z is given implicitly as a function of x, y via the equation $F(x, y, z) = k$, then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for our example at $x = 3$, $y = 2$ and $z > 0$ and note where the formula is valid.

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11.6 Directional Derivative and the Gradient Vector

Let $f(x, y)$ be a function of two variables. We may view f_x and f_y as derivatives in direction of vectors $\mathbf{i}$ and $\mathbf{j}$. This idea may be generalized to any **unit** vector $\mathbf{u}$.

Definition. The directional derivative of f at (x_0, y_0) in direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we set $g(h) = f(x_0 + ha, y_0 + hb)$, then the directional derivative is the same as $g'(0)$, slope of tangent line to the curve shown above.

Note. Why does $\mathbf{u}$ need to be a unit vector?

The function $g(h)$ captures values of f along the line through (x_0, y_0) with direction vector $\mathbf{u}$, parametrized by $\mathbf{r}(h) = \langle x_0 + ha, y_0 + hb\rangle$. Then $g'(h)$ depends on f and on how fast the line is traveled, where speed is $|\mathbf{u}|$. We do not want the directional derivative to depend on the speed the line is traveled, only on its direction, so we set $\mathbf{u}$ to be unit in order to get unit speed.

Theorem. Let $f(x, y)$ be differentiable, and $\mathbf{u} = \langle a, b\rangle$ a unit vector. Then

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b = \nabla f \cdot \mathbf{u},$$

where $\nabla f(x_0, y_0) = f_x(x_0, y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j}$, called the *gradient vector*.

Proof.

Example. Find the directional derivative of $f(x, y) = x^3 + x^2y - y^2$ at point $(2, 1)$ in direction of vector $3\mathbf{i} + 4\mathbf{j}$.

We similarly define the directional derivative of a function of three variables $f(x, y, z)$ in direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$.

Definition. The directional derivative of f at (x_0, y_0, z_0) in direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

As before,

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \mathbf{u},$$

$$\text{where } \nabla f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)\,\mathbf{i} + f_y(x_0, y_0, z_0)\,\mathbf{j} + f_z(x_0, y_0, z_0)\,\mathbf{k}$$

Theorem. For a differentiable function f of two or three variables, the maximum (minimum) value of $D_{\mathbf{u}}f(\mathbf{x})$ over all possible directions $\mathbf{u}$ is $|\nabla f(\mathbf{x})|$ (respectively, $-|\nabla f(\mathbf{x})|$), and it occurs when $\mathbf{u}$ is in the direction of $\nabla f(\mathbf{x})$ (respectively, $-\nabla f(\mathbf{x})$).

Proof.

Applications of gradients

Example. A smelly substance has been smeared throughout the upper half-plane with the function $f(x, y) = x^2 + \frac{1}{y}$ describing its pungency. If you are standing at point $(1, 2)$, in which direction should you run to reduce discomfort the fastest?

Suppose $F(x, y, z) = k$ is a level surface for a function $F(x, y, z)$. Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ be any curve on the level surface. Differentiate the equation below to find the relationship between $\nabla F(\mathbf{r}(t))$ and $\mathbf{r}'(t)$.

$F(x(t), y(t), z(t)) = k$

As before, a tangent plane to the surface at a point contains tangent lines to any curve on the surface going through this point, so the above says that $\nabla F(x_0, y_0, z_0)$ is perpendicular to any tangent line to a curve on the surface through (x_0, y_0, z_0), and is therefore perpendicular to the tangent plane at (x_0, y_0, z_0).

The normal vector of the tangent plane to the surface $F(x, y, z) = k$ at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0)$, so the tangent plane has equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Example. Show that the tangent plane to a sphere at a point is always perpendicular to the radius ending at that point.

The gradient vector points in the direction of the biggest increase of F, which is perpendicular to the level surfaces — of course, because along the level surfaces, the function is constant. Illustrate for function $F(x, y, z) = x^2 + y^2 + z$.

Example. By considering level curves of the pungency function $f(x, y) = x^2 + \dfrac{1}{y}$, draw paths of fastest escape from the smelly substance.

11.7 Maximum and Minimum Values

Let $f(x, y)$ be a function of two variables. We would like to find local and absolute maxima and minima.

Definition. f has a
– local maximum at (a, b) if $f(a, b) \geq f(x, y)$ for all (x, y) in some disk centered at (a, b)
– local minimum at (a, b) if $f(a, b) \leq f(x, y)$ for all (x, y) in some disk centered at (a, b)

Definition. f has an
– absolute maximum at (a, b) if $f(a, b) \geq f(x, y)$ for all (x, y) in domain of f
– absolute minimum at (a, b) if $f(a, b) \leq f(x, y)$ for all (x, y) in domain of f

Suppose f has a local maximum at (a, b):

Theorem. If f has a local maximum or minimum at (a, b) and both $f_x(a, b)$ and $f_y(a, b)$ exist, then

$$f_x(a, b) = 0 \quad f_y(a, b) = 0, \text{ that is, } \nabla f(a, b) = \mathbf{0}.$$

Note. Just knowing that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ does not mean that f has a local maximum or minimum at (a, b). Consider $f(x, y) = y^2 - x^2$ at $(0, 0)$.

Definition. A point (a, b) is called a *critical point* of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of the derivatives does not exist.

From theorem above we know that f will have local maxima and minima at critical points. Once we find the critical points, how to determine what is the behavior at the critical point: local maximum, minimum or saddle point?

Predictably, second derivatives are involved, and we define the expression

$$D(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

Theorem (Second Derivatives Test). Suppose (a, b) is a critical point of f, and second derivatives exist and are continuous on a disk around (a, b).
a) If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).
b) If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).
c) If $D(a, b) < 0$ and $f_{xx}(a, b) < 0$, then f has a saddle point at (a, b).

If $D(a, b) = 0$, the test is inconclusive and other means have to be found.

Example. Find all the local extremes of $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$.

Definition. A set in $\mathbf{R}^2$ or $\mathbf{R}^3$ is *closed* if it contains all its boundary points.

Definition. A set in $\mathbf{R}^2$ or $\mathbf{R}^3$ is *bounded* if it is contained in some disk ($\mathbf{R}^2$) or ball ($\mathbf{R}^3$).

Theorem. If f is continuous on a closed and bounded set D in $\mathbf{R}^2$ (or $\mathbf{R}^3$), then f attains both its absolute minimum and absolute maximum value at some points on D.

To find the absolute minimum and maximum values (and where they occur):
1) Find critical points of f.
2) Parametrize boundary and use it to find critical points of the composite of the function and parametrization.
3) Compare values at all the critical points found.

Example. Find absolute extremes of the function $f(x, y) = 2x^2 - x + y^2 - 2$ on the set $D = \{(x, y) \mid x^2 + y^2 \leq 1,\ x \geq 0\}$.