

Fix the origin and three mutually perpendicular number lines through the origin, called the *axes*, usually marked x , y , z . Usually we make the coordinate system *right-handed*: using your right hand, sweeping from the positive x -axis toward the positive y axis results in the thumb pointing in the direction of the positive z -axis.

Then the correspondence between points in space and triples of numbers is achieved like in the picture:

Pairs of axes span coordinate planes:
the xy -plane, the yz -plane and the xz -plane.
They divide space into 8 octants.

Notation. $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R} = \{(x, y, z) \mid x, y, z \in \mathbf{R}\}$, $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R} = \{(x, y) \mid x, y \in \mathbf{R}\}$
 $\mathbf{R}^3 \longleftrightarrow$ points in space
 $\mathbf{R}^2 \longleftrightarrow$ points in plane

Note.

In a plane, an equation in x , y typically represents a curve.
In space, an equation in x , y and z typically represents a surface.

Example. Sketch the sets with equations $x = 7$ and $x^2 + y^2 = 4$ in the xy -plane.

Example. Sketch the sets with equations $x = 7$ and $x^2 + y^2 = 4$ in space.

We say these sets are cylinders based on a curve in the xy -plane.

Example. Sketch the set with equation $z = y^2$ in space.

Distance formula in three dimensions. Let points be given by coordinates $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. The distance $|P_1P_2|$ between P_1 and P_2 is given by:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Proof is based on the Pythagorean theorem, just like in the plane.

Equation of a sphere with center (h, k, l) and radius r is:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin, the equation is

$$x^2 + y^2 + z^2 = r^2$$

Example. What region is described by the double inequality $4 \leq x^2 + y^2 + z^2 \leq 8$?

Example. What is represented by the equation $x^2 + y^2 + z^2 + 4x - 6y + 2z = 11$?

Many quantities in nature have a direction as well as magnitude (size).

velocity

force exerted on object

displacement

Vectors are used to describe such quantities.

Vector = loosely, a directed line segment, drawn as an arrow

Vectors may be added:

triangle law

parallelogram law

Vectors may be multiplied by a real number (scalar):

Subtracting vectors: may think of it in two ways:

$\mathbf{b} - \mathbf{a}$ is the vector which added to \mathbf{a} gives \mathbf{b}

$$\mathbf{b} - \mathbf{a} = \mathbf{b} + (-1)\mathbf{a}$$

We should be able to add any two vectors, which presents a problem if we say that a vector is a “directed line segment,” since we may easily find two directed line segments that cannot be arranged into the triangle or parallelogram law.

All of these directed line segments are considered “same,” because they have the same direction and magnitude.

More accurately: a vector is a “forest” of directed line segments of same direction and magnitude, while a directed line segment is a “tree” in this forest, said to *represent* the vector.

All of the vectors in picture above have the property: from the starting point the terminal point is reached by going ___ units to right and ___ units down. We can therefore associate vectors in a plane with pairs of numbers, called *components* of the vector.

Definition.

A vector in a plane is an ordered pair of real numbers. $\mathbf{a} = \langle a_1, a_2 \rangle$

A vector in space is an ordered triple of real numbers. $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Geometrically, we can say the vector $\langle a_1, a_2 \rangle$ is the “forest” of directed line segments where the terminal point is always reached from the starting point by going a_1 units right and a_2 units up.

A representation of a vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is a directed line segment \overrightarrow{AB} , where $A = (x, y, z)$ and $B = (x + a_1, x + a_2, x + a_3)$.

If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$, then the directed line segment \overrightarrow{AB} represents the vector

$$\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

If $A = (0, 0, 0)$ and $P = (a_1, a_2, a_3)$, the vector $\overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is called the *position vector* of the point P .

Definition. The length of a vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is the number $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. The zero vector is $\mathbf{0} = \langle 0, 0, 0 \rangle$, the opposite vector of \mathbf{a} is $-\mathbf{a} = (-1)\mathbf{a}$.

Operations on vectors using components

Addition: $\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

Subtraction: $\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$

Multiplication by scalar: $c \langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$ (Note: $|c\mathbf{a}| = |c||\mathbf{a}|$)

Example. For $\mathbf{a} = \langle 2, 1, -1 \rangle$ and $\mathbf{b} = \langle 1, 0, 3 \rangle$ compute:

$$\mathbf{a} + \mathbf{b} =$$

$$\mathbf{a} - \mathbf{b} =$$

$$\frac{1}{2}\mathbf{a} =$$

$$3\mathbf{a} - 5\mathbf{b} =$$

$$|3\mathbf{a} - 5\mathbf{b}| =$$

Properties of vector operations. For all vectors \mathbf{a} , \mathbf{b} and \mathbf{c} and scalars c and d , we have:

$$\begin{array}{llll} \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} & \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} & \mathbf{a} + \mathbf{0} = \mathbf{a} & \mathbf{a} + (-\mathbf{a}) = \mathbf{0} \\ c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b} & (c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a} & (cd)\mathbf{a} = c(d\mathbf{a}) & 1\mathbf{a} = \mathbf{a} \end{array}$$

Proof of some equations.

Set $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.

These vectors are called *standard basis vectors*.

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

Definition. A *unit* vector is any vector whose length is 1. For example, \mathbf{i} , \mathbf{j} and \mathbf{k} are unit vectors. For any vector \mathbf{a} , we can always get a unit vector that is in the same direction as \mathbf{a} , it is simply $\frac{1}{|\mathbf{a}|}\mathbf{a}$.

Example. Find a unit vector in the direction of $\langle 1, 1, -3 \rangle$.

Definition. The *dot product* of vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is the scalar

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Example. $\langle 5, 3, -2 \rangle \cdot \langle -1, 1, 2 \rangle =$

Properties of dot product. For all vectors \mathbf{a} , \mathbf{b} and \mathbf{c} and a scalar c , we have:

$$\begin{aligned} \mathbf{a} \cdot \mathbf{a} &= |\mathbf{a}|^2 & \mathbf{a} \cdot \mathbf{b} &= \mathbf{b} \cdot \mathbf{a} & \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \\ (c\mathbf{a}) \cdot \mathbf{b} &= c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b}) & \mathbf{a} \cdot \mathbf{0} &= 0 \end{aligned}$$

Proof of some equations.

Theorem. If θ is the angle between \mathbf{a} and \mathbf{b} (when positioned tail-to-tail), then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

Proof.

Corollary. If θ is the angle between \mathbf{a} and \mathbf{b} (when positioned tail-to-tail), then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Example. Find the angle between vectors $\langle 3, 1, 1 \rangle$ and $\langle -2, 2, 1 \rangle$.

Note. \mathbf{a} is perpendicular to \mathbf{b} if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Projection of a vector onto another vector.

projection of \mathbf{b} onto \mathbf{a} : $\text{proj}_{\mathbf{a}}\mathbf{b} = (\mathbf{b} \cdot \mathbf{u}) \mathbf{u} = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|^2} \mathbf{a}$ — vector

component of \mathbf{b} in direction \mathbf{a} : $\text{comp}_{\mathbf{a}}\mathbf{b} = \mathbf{b} \cdot \mathbf{u} = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|}$ — scalar
 \pm (length of projection)

(\mathbf{u} is a unit vector in direction of \mathbf{a})

Example. Find the height of the triangle with vertices $A = (0, 2, 0)$, $B = (1, 1, 1)$ and $C = (2, -1, 2)$ from the vertex C .

Work. Suppose a force F acts on an object moving a distance d along a line, and the force is in the direction of motion. Then work is defined as $W = F \cdot d$. What if force is not in direction of motion?

Work done while a constant force \mathbf{F} acts on an object moving along vector \mathbf{D} is

$$W = \mathbf{F} \cdot \mathbf{D}$$

Definition. A *determinant* is a square array of numbers 1×1 , 2×2 , $3 \times 3 \dots$ that represents a single number.

The rules on computing determinants are:

$$1 \times 1: \quad |a| = a, \quad 2 \times 2: \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$3 \times 3: \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ = -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = \dots$$

Determinants with bigger dimensions are defined recursively in a similar way. Signs in the computation follow a checkerboard pattern. Fact: a determinant can be computed by expansion by any row or column and gives the same number.

Example. Expand by third column and third row.

$$\begin{vmatrix} 3 & 1 & -2 \\ 4 & -2 & 0 \\ -1 & 5 & 4 \end{vmatrix} =$$

Facts.

- a) if two rows or columns of a determinant are proportional (equal), the determinant is 0.
- b) swapping rows or columns in a determinant changes the sign of the determinant.

Definition. The *cross product* of vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is the vector

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(Although determinants are defined for real numbers, we extend its use so that a row consists of vectors.)

Example. $\langle 2, 1, 5 \rangle \times \langle 3, -3, 1 \rangle =$

Note. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

It follows that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$
 $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$

Therefore, $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both \mathbf{a} and \mathbf{b} , so it is perpendicular to the plane that \mathbf{a} and \mathbf{b} span.

Geometric interpretation of $\mathbf{a} \times \mathbf{b}$. The vector $\mathbf{a} \times \mathbf{b}$ is the vector that:

- 1) is perpendicular to the plane spanned by \mathbf{a} , \mathbf{b} .
- 2) has direction determined by the right hand rule: as we sweep with the right hand from \mathbf{a} to \mathbf{b} , thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.
- 3) has length $|\mathbf{a}| |\mathbf{b}| \sin \theta$ ($\theta =$ angle between \mathbf{a} , \mathbf{b}).

Note. $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if and only if \mathbf{a} and \mathbf{b} are parallel (that is, $\sin \theta = 0$).

Properties of cross product. For all vectors \mathbf{a} , \mathbf{b} and \mathbf{c} and a scalar c , we have:

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} & (\mathbf{c}\mathbf{a}) \times \mathbf{b} &= c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b}) \\ (\mathbf{a} + \mathbf{b}) \times \mathbf{c} &= \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} & \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} & \mathbf{b} \times \mathbf{a} &= -\mathbf{a} \times \mathbf{b} \end{aligned}$$

Proof of some equations.

Note. There is no associativity: compute $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$ and $\mathbf{i} \times (\mathbf{i} \times \mathbf{j})$

The area of the parallelogram spanned by \mathbf{a} and \mathbf{b} is $|\mathbf{a} \times \mathbf{b}|$.

Example. Find the area of the triangle with vertices $A = (3, 0, 1)$, $B = (1, 1, -2)$ and $C = (-2, 1, 0)$.

Three vectors \mathbf{a} , \mathbf{b} and \mathbf{c} not in the same the plane (not coplanar) span a *parallelepiped* — rectangular box without the right angles. Its volume is given by the *scalar triple product*

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Example. Find the volume of the parallelepiped spanned by $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + \mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j}$.

Note. Vectors \mathbf{a} , \mathbf{b} and \mathbf{c} are coplanar if and only if $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.

Note. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$. Any other order gives you the opposite number.

A line in space is determined by:

- 1) a point P_0 on the line
- 2) a direction vector \mathbf{v}

If \mathbf{r}_0 is the position vector of P_0 , we see the tip of a generic position vector \mathbf{r} will be on the line if and only if there is a number t so that

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

vector equation of the line

Writing the vector equation in components, we get *parametric equations of the line* through point (x_0, y_0, z_0) with direction vector $\langle a, b, c \rangle$

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases} \quad \text{where} \quad \begin{aligned} \mathbf{r} &= \langle x, y, z \rangle \\ \mathbf{r}_0 &= \langle x_0, y_0, z_0 \rangle \\ \mathbf{v} &= \langle a, b, c \rangle \end{aligned}$$

Example. Find the parametric equations of the line through points $A = (3, 1, 0)$ and $B = (-2, 1, -1)$. List a few other points on the line.

Eliminating the parameter from parametric equations of a line we get the

symmetric equations of a line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Write the symmetric equations of the line above.

Example. In the example where direction vector is taken to be \overrightarrow{AB} , for which parameters t do we get only the line segment from A to B ?

Vector equations of a line segment from whose endpoints have position vectors \mathbf{r}_0 and \mathbf{r}_1 :

$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $0 \leq t \leq 1$ and $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$, or $\mathbf{r} = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$, where $0 \leq t \leq 1$

A plane in space is determined by:

- 1) a point P_0 in the plane
- 2) a normal vector \mathbf{n}

If \mathbf{r}_0 is the position vector of P_0 , we see the tip of a generic position vector \mathbf{r} will be on the line if and only if

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$$

vector equation of the plane

Writing the vector equation in components, we get the *scalar equation of the plane* through point (x_0, y_0, z_0) with normal vector $\langle a, b, c \rangle$:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This can also be written in form $ax + by + cz = d$. (Recall that $Ax + By = C$ is the general equation of a line in \mathbf{R}^2 .)

Example. Find the equation of the plane determined by points $P = (-1, 3, 1)$, $Q = (-1, 1, 2)$ and $R = (-2, 3, 2)$. Where does the plane intersect the axes?

To get intercepts, write plane equation in form $\frac{x}{m} + \frac{y}{n} + \frac{z}{p} = 1$. Then the x -, y - and z -intercepts are m , n and p , respectively.

Example. Find the intersection of the line $\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-2}{-1}$ and the plane $2x + 3y - 4z = -2$.

Example. Find the equation of the plane that contains the line $x = 1 + 3t$, $y = 2 + 2t$, $z = 3 + t$ and the point $P = (-1, 0, 1)$.

Example. Find the equation of the line that is the intersection of the planes $2x + y - z = -1$ and $x + 3y + 4z = 2$.

The angle θ between two planes is the *acute* angle between their normal vectors \mathbf{n}_1 and \mathbf{n}_2 .

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|}$$

Recall equations of conic curves:

Parabola

$$y = ax^2$$

Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

Example. Sketch the surface in space whose equation is $\frac{x^2}{4} - \frac{z^2}{9} = 1$.

Quadric surfaces are 3-dimensional analogues of conic curves. To help us see what they look like, we find intersections of the surfaces with coordinate planes, or planes parallel to coordinate planes (*traces or cross-sections* of the surface).

Example. Sketch the surface given by the equation $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1$.

Example. Sketch the surface given by the equation $\frac{x^2}{16} + \frac{y^2}{4} - \frac{z^2}{25} = 1$.

Example. Sketch the surface given by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$.

There are six types of quadric surfaces — remember their shapes, so you can guess what the surface is, based on traces in coordinate planes or planes parallel to coordinate planes.

The general equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0,$$

where at least one of A – F is nonzero, has a graph that is:

- typically one of the quadric surfaces, possibly rotated
- a cylinder based on a conic curve, possibly rotated
- less commonly, nothing, a point or one or two planes

Note the left side is a general polynomial of degree 2 in variables x, y, z .

The six quadric surfaces:

Ellipsoid

Cone

Elliptic Paraboloid

Hyperboloid of One Sheet

Hyperbolic Paraboloid
(saddle surface)

Hyperboloid of Two Sheets

Definition. A *vector function* is a function whose domain is a set of real numbers, and range is a set of vectors (two- or three-dimensional).

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t) \mathbf{i} + g(t) \mathbf{j} + h(t) \mathbf{k}$$

Note that, as t varies, the tips of the position vector $\mathbf{r}(t)$ trace out a space curve. We say that $x = f(t)$, $y = g(t)$, $z = h(t)$ are *parametric equations* of this curve.

Example. Sketch the curve $\mathbf{r}(t) = \langle 3 \cos t, 5t, 3 \sin t \rangle$, $0 \leq t \leq 6\pi$.

Example. $\mathbf{r}(t) = \mathbf{r}_0(t) + t\mathbf{v}$ parametrizes a line through tip of \mathbf{r}_0 with direction vector \mathbf{v} .

Example. Parametrize the line segment from $A = (1, -2, 1)$ to $B = (2, 3, 0)$. Then give another parametrization of it.

Definition. The limit of a vector function is $\lim_{t \rightarrow a} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle$.

Definition. A vector function is continuous if $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a)$.

Example. A vector function is continuous if the curve given by $\mathbf{r}(t)$ has no breaks. If every component of the vector function is a single formula, the vector function is continuous wherever all the component functions are defined.

Definition. The *derivative* of a vector function $\mathbf{r}(t)$ is defined as

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists.

Meaning of $\mathbf{r}'(t)$:

As $h \rightarrow 0$, the direction of \overrightarrow{PQ} appears to get closer and closer to the direction of the tangent line. However, $\overrightarrow{PQ} \rightarrow 0$, so in order to get a nonzero vector, we divide by h , which is proportional to length of \overrightarrow{PQ} for small h .

Note. The *unit tangent vector* is $\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

Theorem. If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$.

Proof.

Differentiation rules. For vector functions $\mathbf{u}(t)$, $\mathbf{v}(t)$ and scalar function $f(t)$ we have:

$$\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$$

$$\frac{d}{dt}(c\mathbf{u}(t)) = c\mathbf{u}'(t)$$

$$\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

$$\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$$

$$\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

$$\frac{d}{dt}\mathbf{u}(f(t)) = \mathbf{u}'(f(t))f'(t) = f'(t)\mathbf{u}'(f(t))$$

Definition. The *definite integral* of a vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is defined as

$$\int_a^b \mathbf{r}(t) dt = \left\langle \int_a^b f(t) dt, \int_a^b g(t) dt, \int_a^b h(t) dt \right\rangle$$

(We will not be using this type of integral a whole lot.)

Theorem. The length of a space curve parametrized by $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$ is given by

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt = \int_a^b |\mathbf{r}'(t)| dt$$

Example. Compute the length of the curve $x = \cos(4t)$, $y = 2t^{\frac{3}{2}}$, $z = \sin(4t)$, $0 \leq t \leq \pi$.

It would be nice if the parameter also indicated how long the curve was up to that point.

Introduce the *arclength function* $s(t) = \int_a^t |\mathbf{r}'(u)| du$ which represents the length of the curve from $u = a$ to $u = t$.

Since $s(t)$ is an increasing function of t , it has an inverse $t(s)$. If we use s as a new parameter, the length of the curve $\mathbf{r}(t(s))$ from $s = 0$ to $s = s_0$ will be exactly s_0 .

Example. Reparametrize the above example with respect to arclength from $t = 0$ in direction of increasing t .

In calculus 1, we defined

$$\text{velocity} = \lim_{h \rightarrow 0} \frac{\text{displacement from } t \text{ to } t+h}{\text{change in time}} = \left[\begin{array}{l} \text{Generalize to} \\ \text{3-dimensional} \\ \text{motion} \end{array} \right] = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t)$$

Therefore, if $\mathbf{r}(t)$ represents motion of a particle, it is reasonable to define:

$$\text{velocity} = \mathbf{v}(t) = \mathbf{r}'(t) \qquad \text{acceleration} = \mathbf{a}(t) = \mathbf{v}'(t)$$

speed (a scalar) is defined as $|\mathbf{v}(t)|$

Example. Let $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$. Compute $\mathbf{v}(t)$ and $\mathbf{a}(t)$. Draw and comment.

Note. $\mathbf{v}(t)$ is tangent to the curve.

$\mathbf{a}(t)$ shows which way the particle turns to stay on the curve.

$$|\mathbf{v}(t)| = 1 \text{ for every } t$$

What happens to $\mathbf{v}(t)$ if the curve is traversed twice as fast, that is $\mathbf{r}(t) = \langle \cos(2t), \sin(2t), 0 \rangle$?

Example. Suppose $\mathbf{a}(t) = \langle \sin t, 2, 2e^{2t} \rangle$. Find the position $\mathbf{r}(t)$ if we know that $\mathbf{r}(0) = \langle 0, 1, -4 \rangle$ and $\mathbf{v}(0) = \langle 4, 0, 2 \rangle$.

From this problem, we see: if force at every time is known on an object (hence $\mathbf{a}(t)$ is known), then position is known at every time. This problem is usually called the *initial value problem*.

Example. If $|\mathbf{v}(t)|$ is constant, then $\mathbf{a}(t)$ is perpendicular to $\mathbf{v}(t)$. (Note in this case force does not do any work).