Matrix Theory — Exam 1
MAT 335, Fall 2022 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (12pts) For the matrices A, B and C find the following expressions, if they are defined:
a) AA^T b) BA c) $CB - B$

$$A = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 5 \\ 2 & -1 \end{bmatrix}$$

2. (8pts) The augmented matrix below is in reduced row echelon form.
a) Write the solution of the system represented by the matrix.
b) Write the solution in vector form.

$$A = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 3 & 4 \end{array}\right]$$

3. (8pts) Write the rotation matrix for a counterclockwise rotation around the origin by angle $\frac{2\pi}{3}$ and use it find where the point $(2, 1)$ lands after it is rotated.

4. (14pts) A system of linear equations is given below.
a) Use the Gaussian elimination to solve the system.
b) Write the solution in vector form.

$$\begin{cases} 3x_1 & -2x_2 & -3x_3 & -15x_4 & = & -9 \\ x_1 & -x_2 & -2x_3 & -8x_4 & = & -3 \\ -2x_1 & +x_2 & +2x_3 & +9x_4 & = & 5 \end{cases}$$

5. (10pts) Below is the input-output matrix for an economy producing food, transportation and tourism.
a) What net production corresponds to a gross production of \$30M of food, \$50M of transportation and \$40M of tourism?
b) Set up the system (write its augmented matrix) to find the gross production needed to satisfy exactly demand of \$35M of food, \$20M of transportation and \$50M of tourism. Do NOT solve the system, or you will suffer.

food	transp.	tourism	
0.2	0.2	0.3	food
0.3	0.1	0.3	transportation
0.1	0.1	0.1	tourism

6. (12pts) Below is the augmented matrix of a system of linear equations. Determine the coefficients a and b for which the system has: a) one solution, b) infinitely many solutions, c) no solutions. (Note: no row operations are needed.)

$$A = \left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 0 & a & -2 & 3 \\ 0 & 0 & b+3 & 7 \end{array}\right]$$

7. (6pts) Find the elementary matrix E so that $EA = B$.

$$A = \begin{bmatrix} 0 & -2 & 7 \\ 3 & 2 & -1 \\ -4 & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -2 & 7 \\ -4 & 6 & 0 \end{bmatrix} \qquad E =$$

8. (12pts) Consider the vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ -3 \end{bmatrix}$.

a) Do they span $\mathbf{R}^4$?
b) Are they linearly independent?

9. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If $\mathbf{a}$ and $\mathbf{b}$ are in Span$\{\mathbf{u}, \mathbf{v}\}$, then $\mathbf{a} - 3\mathbf{b}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$.
b) If A is a 3×3 matrix with rank 2, then the columns of A are linearly dependent.
c) If 2×2 matrices A and B are invertible, then $A + B$ is invertible.

Bonus. (10pts) Show: if vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, then vectors $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are linearly independent.

Matrix Theory — Exam 2
MAT 335, Fall 2022 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (12pts) Matrix A is given below.
a) Evaluate its determinant by any (efficient) method.
b) State if A is invertible and justify.

$$\begin{vmatrix} 1 & 3 & 0 & 0 \\ -1 & 1 & 2 & -3 \\ 2 & 6 & 3 & 0 \\ 0 & 4 & 2 & 1 \end{vmatrix} =$$

2. (6pts) If A and B are 3×3 matrices with $\det A = 3$ and $\det B = -2$, find:

$\det(A^{-1}B) =$

$\det(2A) =$

$\det(-A^T B) =$

3. (12pts) The matrix A is given below.
a) Find the inverse of A.
b) Use the inverse to easily solve the system below.

$$A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

$$\begin{aligned} 3x_1 &\ +5x_2 &= -2 \\ x_1 &\ +2x_2 &= 1 \end{aligned}$$

4. (14pts) Find the standard matrix of the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ and determine whether T is a) one-to-one, or b) onto.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - x_2 + 4x_3 \\ 2x_1 + x_2 + x_3 \\ 3x_1 + 9x_2 - 6x_3 \end{bmatrix}$$

5. (8pts) For a function $T : \mathbf{R}^2 \to \mathbf{R}^3$ the following is known:
a) T is a linear transformation
b) $T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

Find the standard matrix of T.

6. (16pts) A matrix A is given below.
a) Find a basis for the nullspace of A.
b) Find a basis for the column space of A.

$$A = \begin{bmatrix} 1 & 4 & 1 & -1 \\ -2 & -7 & 0 & -1 \\ 1 & 5 & 3 & -3 \end{bmatrix}$$

7. (14pts) The set W is defined below.
a) Use the definition to show W is a subspace of $\mathbf{R}^3$.
b) Give a set of generating vectors for W.

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3 \mid x_1 + 2x_2 = 0 \text{ and } x_2 - x_3 = 0 \right\}$$

8. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.

a) If $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent and T is a *nonzero* linear transformation, then $\{T(\mathbf{u}_1), T(\mathbf{u}_2)\}$ is linearly independent.

b) For a 2×2 matrix A, if $\det A = 0$, then the columns of A are parallel.

c) The set $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbf{R}^2 \mid x_1^2 + x_2^2 \geq 1 \right\}$ is a subspace of $\mathbf{R}^2$.

Bonus. (10pts) Let A be an $n \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$. If B is the matrix with columns $\mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_n, \mathbf{a}_1$ (in that order), how are $\det A$ and $\det B$ related?

Matrix Theory — Exam 3
MAT 335, Fall 2022 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (12pts) Give a basis for the row space of A and state the dimension of Null A.

$$A = \begin{bmatrix} 2 & 5 & -3 & 0 \\ 1 & 3 & 5 & -4 \\ 7 & 19 & 9 & -12 \end{bmatrix}$$

2. (8pts) One of the vectors is an eigenvector for the matrix A below. Determine which one, and the eigenvalue it corresponds to.

$$A = \begin{bmatrix} 2 & -6 & 6 \\ 1 & 9 & -6 \\ -2 & 16 & -13 \end{bmatrix} \quad \text{vectors: } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

3. (14pts) Let W be the subspace of $\mathbf{R}^4$ spanned by the set given below. Find a basis for $W^{\perp}$.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

4. (16pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find the basis of the corresponding eigenspace.

$$A = \begin{bmatrix} -6 & -8 & 0 \\ 3 & 4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

5. (12pts) a) Verify that the vectors at left are an orthonormal set.
b) The vector at right is in the subspace spanned by the vectors. Write it as a linear combination of the vectors from the orthonormal set. (Avoid solving a system: use the fact the set is orthonormal.)

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right\} \qquad \begin{bmatrix} 7 \\ 6 \\ 2 \\ 1 \end{bmatrix}$$

6. (10pts) A 3×3 matrix A has eigenvalues 1, and -3, and the dimension of the eigenspace corresponding to eigenvalue 1 is 2.
a) Determine the characteristic polynomial of A and justify.
b) Use the characteristic polynomial to evaluate $\det(A - 5I)$.

7. (10pts) Show: the vectors $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal if and only if $|\mathbf{u}| = |\mathbf{v}|$. (Do not use coordinates. I beseech you.)

8. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If 0 is an eigenvalue of A, then A is not invertible.
b) If the characteristic polynomial of a 2×2 matrix A is $(t-3)^2$, then $A = 3I$.
c) If A is 2×2 matrix, then $(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for every $\mathbf{x}$, $\mathbf{y}$ in $\mathbf{R}^2$.

Bonus. (10pts) An $n \times n$ matrix A is called orthogonal if $A^T A = I_n$. Prove the following statements.
a) A 2×2 rotation matrix is orthogonal.
b) The columns of A are an orthonormal basis for $\mathbf{R}^n$.
c) Multiplying by A preserves the norm of a vector, that is, $|A\mathbf{x}| = |\mathbf{x}|$.
(Hint: show $|A\mathbf{x}|^2 = |\mathbf{x}|^2$.)

Matrix Theory — Final Exam **MAT 335, Fall 2022 — D. Ivanšić**	**Name:** ____________________ *Show all your work!*

1. (12pts) For the matrices A, B and C find the following expressions, if they are defined:
a) $CA + B$ b) $B^T C$ c) $A^T C C^T$

$$A = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -7 & -1 \\ 3 & 0 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 & 4 \\ -3 & 1 \end{bmatrix}$$

2. (18pts) A matrix A is given at right.
a) Find a basis for the nullspace of A.
b) Find a basis for the column space of A.
c) Find a basis for the row space of A.
d) Among $\operatorname{Row} A$, $\operatorname{Col} A$ and $\operatorname{Null} A$, are any two orthogonal, and if so, which ones?
(Note: only one Gauss elimination method is needed.)

$$A = \begin{bmatrix} 1 & -1 & -3 & 1 \\ -2 & 1 & 5 & 0 \\ 4 & -2 & -10 & 1 \end{bmatrix}$$

3. (12pts) Below is the augmented matrix of a system of linear equations. Determine the coefficient b for which the system has: a) one solution, b) infinitely many solutions, c) no solutions. (Note: no row operations are needed.)

$$A = \left[\begin{array}{ccc|c} 1 & -2 & 7 & 1 \\ 0 & b & 0 & 4 \\ 0 & 0 & b^2+b & b \end{array}\right]$$

4. (10pts) Consider the vectors at right.
a) Are they linearly independent?
b) Are they orthogonal?
c) Are they a basis for $\mathbf{R}^n$?

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

5. (12pts) Matrix A is given below.
a) Evaluate its determinant by any (efficient) method.
b) State if A is invertible and justify.

$$\begin{vmatrix} 0 & 3 & 0 & 1 \\ -1 & 4 & 2 & -3 \\ 2 & -2 & 3 & -2 \\ -3 & 9 & 2 & 1 \end{vmatrix} =$$

6. (6pts) Write the rotation matrix for a counterclockwise rotation around the origin by angle $\frac{3\pi}{4}$ and use it find where the point $(-1, 3)$ lands after it is rotated.

7. (12pts) The matrix A is given below.
a) Find the inverse of A.
b) Use the inverse to easily solve the system below.

$$A = \begin{bmatrix} -2 & 3 \\ -1 & 5 \end{bmatrix}$$

$$\begin{array}{rrl} -2x_1 & +3x_2 & = 3 \\ -x_1 & +5x_2 & = -4 \end{array}$$

8. (10pts) Find the standard matrix of the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ and determine whether T is a) one-to-one, or b) onto.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + 4x_2 \\ x_1 - 3x_2 \\ 2x_1 + 4x_2 \end{bmatrix}$$

9. (18pts) The set W is defined below.
a) Use the definition to show W is a subspace of $\mathbf{R}^3$.
b) Give a set of generating vectors for W.
c) Find a basis for $W^\perp$.

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3 \mid x_1 - x_2 + 2x_3 = 0 \right\}$$

10. (24pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent and T is a *nonzero* linear transformation, then $\{T(\mathbf{u}_1), T(\mathbf{u}_2)\}$ is linearly independent.
b) If A is a 4×3 matrix with rank 2, then the columns of A are linearly dependent.
c) If the characteristic polynomial of a 2×2 matrix A is $(t-3)^2$, then $A = 3I$.
Justify your answer by giving a logical argument or a counterexample.
d) If -2 and 3 are eigenvalues of a 2×2 matrix A, then A is invertible.

11. (16pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find the basis of the corresponding eigenspace.

$$A = \begin{bmatrix} -3 & 3 & 9 \\ 0 & 3 & 0 \\ -2 & 1 & 6 \end{bmatrix}$$

Bonus. (10pts) Show: if vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, then vectors $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are linearly independent.