Matrix Theory — Exam 3 **MAT 335, Fall 2022 — D. Ivanšić**	**Name:** Saul Ocean *Show all your work!*

1. (12pts) Give a basis for the row space of A and state the dimension of Null A.

$$A = \begin{bmatrix} 2 & 5 & -3 & 0 \\ 1 & 3 & 5 & -4 \\ 7 & 19 & 9 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -4 \\ 2 & 5 & -3 & 0 \\ 7 & 19 & 9 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -4 \\ 0 & -1 & -13 & 8 \\ 0 & -2 & -26 & 16 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & -4 \\ 0 & 1 & 13 & -8 \\ \end{bmatrix}$$ ← these rows are lin. indep. so are a basis for Row A

dim Null A = nullity A = $4 - \text{rank } A = 4 - 2 = 2$

2. (8pts) One of the vectors is an eigenvector for the matrix A below. Determine which one, and the eigenvalue it corresponds to.

$$A = \begin{bmatrix} 2 & -6 & 6 \\ 1 & 9 & -6 \\ -2 & 16 & -13 \end{bmatrix} \quad \text{vectors: } \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}}$$

$A\vec{v}_1 = \begin{bmatrix} -4 \\ 7 \\ 9 \end{bmatrix}$ $\quad A\vec{v}_2 = \begin{bmatrix} 4 \\ -4 \\ -8 \end{bmatrix} = -4\vec{v}_2$

$\vec{v}_2$ is an eigenvector with eigenvalue -4

3. (14pts) Let W be the subspace of $\mathbf{R}^4$ spanned by the set given below. Find a basis for $W^\perp$.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 2 & -2 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$x_1 = x_4$
$x_2 = -x_4$
$x_3 = x_4$
x_4 free

$\vec{x} = x_4 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is basis for $W^\perp$

4. (16pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find the basis of the corresponding eigenspace.

$$A = \begin{bmatrix} -6 & -8 & 0 \\ 3 & 4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det(A-tI) = \begin{vmatrix} -6-t & -8 & 0 \\ 3 & 4-t & 0 \\ 0 & 1 & -t \end{vmatrix} = -t\left((-6-t)(4-t)+24\right)$$

expand by 3rd column

$$= -t\left((t+6)(t-4)+24\right) = -t\left(t^2+2t-24+24\right)$$

$$= -t^2(t+2)$$

$-t^2(t+2)=0$

$t=0,-2$

$$A-0I = \begin{bmatrix} -6 & -8 & 0 \\ 3 & 4 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 3 & 4 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$x_1=0$
$x_2=0$
x_3 free

$$\vec{x} = x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

basis: $\vec{e}_3$

$$A+2I = \begin{bmatrix} -4 & -8 & 0 \\ 3 & 6 & 0 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \end{bmatrix}$$

$x_1 = 4x_3$
$x_2 = -2x_3$

$$\vec{x} = x_3\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

basis for eigenspace corresp. to -2

5. (12pts) a) Verify that the vectors at left are an orthonormal set.
b) The vector at right is in the subspace spanned by the vectors. Write it as a linear combination of the vectors from the orthonormal set. (Avoid solving a system: use the fact the set is orthonormal.)

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right\} \qquad \vec{x} = \begin{bmatrix} 7 \\ 6 \\ 2 \\ 1 \end{bmatrix}$$

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

$\vec{v}_1 \cdot \vec{v}_2 = 0 - \frac{1}{3} + \frac{1}{3} + 0 = 0$

$\vec{v}_2 \cdot \vec{v}_3 = 0 + 0 - \frac{1}{3} + \frac{1}{3} = 0$

$\vec{v}_3 \cdot \vec{v}_1 = \frac{1}{3} + 0 - \frac{1}{3} + 0 = 0$

$|\vec{v}_1| = \sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+0} = \sqrt{1} = 1$

$|\vec{v}_2| = \sqrt{0+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = \sqrt{1} = 1$

$|\vec{v}_3| = \sqrt{\frac{1}{3}+0+\frac{1}{3}+\frac{1}{3}} = \sqrt{1} = 1$

Because vectors are orthonormal, coefficients are

$\vec{x}\cdot\vec{v}_1 = \frac{7}{\sqrt{3}} + \frac{6}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{15}{\sqrt{3}} = 5\sqrt{3}$

$\vec{x}\cdot\vec{v}_2 = -\frac{6}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$

$\vec{x}\cdot\vec{v}_3 = \frac{7}{\sqrt{3}} - \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$

$\vec{x} = 5\sqrt{3}\,\vec{v}_1 - \sqrt{3}\,\vec{v}_2 + 2\sqrt{3}\,\vec{v}_3$

6. (10pts) A 3×3 matrix A has eigenvalues 1, and -3, and the dimension of the eigenspace corresponding to eigenvalue 1 is 2.
a) Determine the characteristic polynomial of A and justify.
b) Use the characteristic polynomial to evaluate $\det(A - 5I)$.

a) char. poly is $(t-1)^a(t+3)^b(\,?\,)$

$a \geq 2 \quad b \geq 1$ and $a+b \leq 3$

The only possibility is $a=2$ and $b=1$, and no other factor, since degree is 3

$\det(A-tI) = -(t-1)^2(t+3)$

b) $\det(A-5I) = -(5-1)^2(5+3) = -16 \cdot 8 = -128$

7. (10pts) Show: the vectors $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal if and only if $|\mathbf{u}| = |\mathbf{v}|$. (Do not use coordinates. I beseech you.)

$(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = 0$

$\vec{u}\cdot\vec{u} + \underbrace{\vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v}}_{=0} - \vec{v}\cdot\vec{v} = 0$

$|\vec{u}|^2 - |\vec{v}|^2 = 0$

$|\vec{u}|^2 = |\vec{v}|^2$

$|\vec{u}| = |\vec{v}|$

All steps are reversible

so $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = 0 \Leftrightarrow |\vec{u}| = |\vec{v}|$

8. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.

a) If 0 is an eigenvalue of A, then A is not invertible.

b) If the characteristic polynomial of a 2×2 matrix A is $(t-3)^2$, then $A = 3I$.

c) If A is 2×2 matrix, then $(A\mathbf{x})\cdot(A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for every $\mathbf{x}$, $\mathbf{y}$ in $\mathbf{R}^2$.

a) True, If 0 is an eigenvalue, then $A\vec{x} = 0\vec{x} = \vec{0}$ for some nonzero $\vec{x}$.

But then A is not invertible.

b) False. Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ $\det(A - tI) = \begin{vmatrix} 3-t & 1 \\ 0 & 3-t \end{vmatrix} = (3-t)^2 = (t-3)^2$ but $A \neq 3I$

c) False. Let $A = 2I$. $(2I\vec{x})\cdot(2I\vec{y}) = (2\vec{x})\cdot(2\vec{y}) = 4\vec{x}\cdot\vec{y} \neq \vec{x}\cdot\vec{y}$

Bonus. (10pts) An $n \times n$ matrix A is called orthogonal if $A^T A = I_n$. Prove the following statements.

a) A 2×2 rotation matrix is orthogonal.

b) The columns of A are an orthonormal basis for $\mathbf{R}^n$.

c) Multiplying by A preserves the norm of a vector, that is, $|A\mathbf{x}| = |\mathbf{x}|$.
(Hint: show $|A\mathbf{x}|^2 = |\mathbf{x}|^2$.*)*

a) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) The ij-th element of $A^T A$ is $(i\text{-th column of } A)\cdot(j\text{-th column of } A) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

since $A^T A = I$

This means that columns are orthogonal and have norm 1

c) $|A\vec{x}|^2 = (A\vec{x})\cdot(A\vec{x}) = \vec{x}\cdot \underbrace{A^T A}_{=I} \vec{x} = \vec{x}\cdot\vec{x} = |\vec{x}|^2$

$I\vec{x} = \vec{x}$