Matrix Theory — Exam 2
MAT 335, Fall 2022 — D. Ivanšić

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Show all your work!

1. (12pts) Matrix A is given below.
a) Evaluate its determinant by any (efficient) method.
b) State if A is invertible and justify.

$$\begin{vmatrix} 1 & 3 & 0 & 0 \\ -1 & 1 & 2 & -3 \\ 2 & 6 & 3 & 0 \\ 0 & 4 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 0 \\ 0 & 4 & 2 & -3 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 2 & 1 \end{vmatrix} = \text{expand by 1st column} = 1 \cdot \begin{vmatrix} 4 & 2 & -3 \\ 0 & 3 & 0 \\ 4 & 2 & 1 \end{vmatrix} = \text{expand by 2nd row}$$

$$= 3 \cdot \begin{vmatrix} 4 & -3 \\ 4 & 1 \end{vmatrix} = 3(4+12) = 48$$

b) Since $\det A \neq 0$, A is invertible

2. (6pts) If A and B are 3×3 matrices with $\det A = 3$ and $\det B = -2$, find:

$\det(A^{-1}B) = \det A^{-1} \det B = \frac{1}{\det A} \det B = \frac{1}{3} \cdot (-2) = -\frac{2}{3}$

$\det(2A) = 2^3 \det A = 8 \cdot 3 = 24$

$\det(-A^T B) = (-1)^3 \det A^T \det B = -\det A \det B = -3 \cdot (-2) = 6$

3. (12pts) The matrix A is given below.
a) Find the inverse of A.
b) Use the inverse to easily solve the system below.

$$A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

$$\begin{aligned} 3x_1 &+5x_2 &= -2 \\ x_1 &+2x_2 &= 1 \end{aligned}$$

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & 5 & 1 & 0 \end{array}\right] \cdot(-3) \sim \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{array}\right] \cdot 2$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

Solution is $A^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \end{bmatrix}$

4. (14pts) Find the standard matrix of the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ and determine whether T is a) one-to-one, or b) onto.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - x_2 + 4x_3 \\ 2x_1 + x_2 + x_3 \\ 3x_1 + 9x_2 - 6x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 1 \\ 3 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 1 \\ 3 & 9 & -6 \end{bmatrix} \begin{matrix} \cdot(-2) \\ \\ \cdot(-3) \end{matrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -5 \\ 0 & 15 & -15 \end{bmatrix} \cdot(-3)$$

$$\sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

rank = 2 < no of rows, so T not onto

nullity = 1 so T is not one-to-one

5. (8pts) For a function $T : \mathbf{R}^2 \to \mathbf{R}^3$ the following is known:

a) T is a linear transformation

b) $T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

Find the standard matrix of T.

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = 3\vec{e}_1 - \vec{e}_2$$

$$T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = T(3\vec{e}_1) - T(\vec{e}_2)$$

$$\begin{bmatrix} -1 \\ 4 \end{bmatrix} = 3T(\vec{e}_1) - \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$3T(\vec{e}_1) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

$$T(\vec{e}_1) = \begin{bmatrix} -\frac{4}{3} \\ 2 \end{bmatrix}$$

Stand matrix $= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$

$$= \begin{bmatrix} -\frac{4}{3} & -3 \\ 2 & 2 \end{bmatrix}$$

6. (16pts) A matrix A is given below.
a) Find a basis for the nullspace of A.
b) Find a basis for the column space of A.

$$A = \begin{bmatrix} 1 & 4 & 1 & -1 \\ -2 & -7 & 0 & -1 \\ 1 & 5 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 1 & -1 \\ 0 & 1 & 2 & -3 \\ 0 & 1 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 1 & -1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_1 = 7x_3$
$x_2 = -2x_3$
$x_4 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 7 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$ is basis for Null A

Pivot columns are basis for Col A $= \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix} \right\}$

7. (14pts) The set W is defined below.
a) Use the definition to show W is a subspace of $\mathbf{R}^3$.
b) Give a set of generating vectors for W.

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3 \mid x_1 + 2x_2 = 0 \text{ and } x_2 - x_3 = 0 \right\}$$

Let $\vec{x}, \vec{y} \in W$. Then $x_1 + 2x_2 = 0$, $x_2 - x_3 = 0$
$y_1 + 2y_2 = 0$, $y_2 - y_3 = 0$

Check condition on comp. of $\vec{x} + \vec{y}$:

$x_1 + y_1 + 2(x_2 + y_2) = x_1 + 2x_2 + y_1 + 2y_2 = 0 + 0 = 0$

$x_2 + y_2 - (x_3 + y_3) = x_2 - x_3 + y_2 - y_3 = 0 + 0 = 0$

so $\vec{x} + \vec{y} \in W$

$cx_1 + 2cx_2 = c(x_1 + 2x_2) = c \cdot 0 = 0$

$cx_2 - cx_3 = c(x_2 - x_3) = c \cdot 0 = 0$

so $c\vec{x} \in W$

b) $x_1 = -2x_2$
$x_3 = x_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Generated by $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

8. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.

a) If $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent and T is a *nonzero* linear transformation, then $\{T(\mathbf{u}_1), T(\mathbf{u}_2)\}$ is linearly independent.

b) For a 2×2 matrix A, if $\det A = 0$, then the columns of A are parallel.

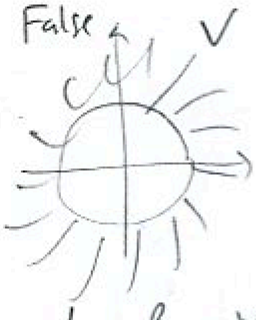
c) The set $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbf{R}^2 \mid x_1^2 + x_2^2 \geq 1 \right\}$ is a subspace of $\mathbf{R}^2$.

a) False: Take $\underbrace{\vec{u}_1 = \vec{e}_1, \vec{u}_2 = \vec{e}_2}_{\text{lin. indep.}}$ $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. $\{A\vec{e}_1, A\vec{e}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

not lin. indep.

b) True. If $\det A = 0$, A is not invertible so columns are linearly dependent. In the case of two vectors, this means they are parallel

c) False



not a line nor plane, not a subspace

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in V$ but $\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \notin V$

$1^2 + 1^2 \geq 1$

since $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 < 1$

Bonus. (10pts) Let A be an $n \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$. If B is the matrix with columns $\mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_n, \mathbf{a}_1$ (in that order), how are $\det A$ and $\det B$ related?

$$\det A = \left| \vec{a}_1 \; \vec{a}_2 \ldots \vec{a}_{n-1} \; \vec{a}_n \right| = (-1) \left| \vec{a}_2 \; \vec{a}_1 \; \vec{a}_3 \ldots a_{n-1} \; \vec{a}_n \right| = (-1)(-1) \left| \vec{a}_2 \; \vec{a}_3 \; \vec{a}_1 \; \vec{a}_4 \ldots \vec{a}_n \right|$$

$$= \text{etc} = (-1)^{n-1} \left| \vec{a}_2 \; \vec{a}_3 \ldots \vec{a}_n \; \vec{a}_1 \right) = (-1)^{n-1} \det B$$

$\vec{a}_1$ swapped places with $\vec{a}_2, \vec{a}_3, \ldots \vec{a}_n$, $n-1$ swaps