Matrix Theory — Exam 1
MAT 335, Fall 2022 — D. Ivanšić

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Show all your work!

1. (12pts) For the matrices A, B and C find the following expressions, if they are defined:
a) AA^T b) BA c) $CB - B$

$$A = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 5 \\ 2 & -1 \end{bmatrix}$$

a) $AA^T = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \begin{bmatrix} 3 & -1 \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$

b) BA is $(2\times 3)(2\times 1)$ not defined

not same

c) $\begin{bmatrix} 0 & 5 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 4 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix}$

$= \begin{bmatrix} 0 & -10 & 10 \\ 8 & 4 & -4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -11 & 11 \\ 8 & 6 & -6 \end{bmatrix}$

2. (8pts) The augmented matrix below is in reduced row echelon form.
a) Write the solution of the system represented by the matrix.
b) Write the solution in vector form.

$$A = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 3 & 4 \end{array}\right]$$

$x_1 \quad + 2x_4 = -1$

$x_3 + 3x_4 = 4$

$x_1 = -1 - 2x_4$

$x_3 = 4 - 3x_4$

x_2, x_4 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

3. (8pts) Write the rotation matrix for a counterclockwise rotation around the origin by angle $\frac{2\pi}{3}$ and use it find where the point $(2, 1)$ lands after it is rotated.

$$A_{\frac{2\pi}{3}} = \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 - \frac{\sqrt{3}}{2} \\ \sqrt{3} - \frac{1}{2} \end{bmatrix}$$

4. (14pts) A system of linear equations is given below.
a) Use the Gaussian elimination to solve the system.
b) Write the solution in vector form.

$$\begin{cases} 3x_1 - 2x_2 - 3x_3 - 15x_4 = -9 \\ x_1 - x_2 - 2x_3 - 8x_4 = -3 \\ -2x_1 + x_2 + 2x_3 + 9x_4 = 5 \end{cases} \sim \left[\begin{array}{cccc|c} 1 & -1 & -2 & -8 & -3 \\ 3 & -2 & -3 & -15 & -9 \\ -2 & 1 & 2 & 9 & 5 \end{array}\right] \begin{array}{l} \cdot(-3) \\ \cdot 2 \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -8 & -3 \\ 0 & 1 & 3 & 9 & 0 \\ 0 & -1 & -2 & -7 & -1 \end{bmatrix} \overset{+}{\sim} \begin{bmatrix} 1 & -1 & -2 & -8 & -3 \\ 0 & 1 & 3 & 9 & 0 \\ 0 & 0 & 1 & 2 & -1 \end{bmatrix} \begin{array}{l} \cdot 2 \\ \cdot(-3) \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & -4 & -5 \\ 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & 2 & -1 \end{bmatrix} \overset{+}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$x_1 = -2 + x_4$$
$$x_2 = 3 - 3x_4$$
$$x_3 = -1 - 2x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -3 \\ -2 \\ 1 \end{bmatrix}$$

5. (10pts) Below is the input-output matrix for an economy producing food, transportation and tourism.
a) What net production corresponds to a gross production of \$30M of food, \$50M of transportation and \$40[redacted]M of tourism?
b) Set up the system (write its augmented matrix) to find the gross production needed to satisfy exactly demand of \$35M of food, \$20M of transportation and \$50M of tourism. Do NOT solve the system, or you will suffer.

food	transp.	tourism	
0.2	0.2	0.3	food
0.3	0.1	0.3	transportation
0.1	0.1	0.1	tourism

a) $$(I-C)\begin{bmatrix} 30 \\ 50 \\ 40 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.2 & -0.3 \\ -0.3 & 0.9 & -0.3 \\ -0.1 & -0.1 & 0.9 \end{bmatrix} \begin{bmatrix} 30 \\ 50 \\ 40 \end{bmatrix} = \begin{bmatrix} 24 - 10 - 12 \\ -9 + 45 - 12 \\ -3 - 5 + 36 \end{bmatrix} = \begin{bmatrix} 2 \\ 24 \\ 28 \end{bmatrix}$$

b) $$\left[\begin{array}{ccc|c} 0.8 & -0.2 & -0.3 & 35 \\ -0.3 & 0.9 & -0.3 & 20 \\ -0.1 & -0.1 & 0.9 & 50 \end{array}\right]$$

6. (12pts) Below is the augmented matrix of a system of linear equations. Determine the coefficients a and b for which the system has: a) one solution, b) infinitely many solutions, c) no solutions. (Note: no row operations are needed.)

$$A = \left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 0 & a & -2 & 3 \\ 0 & 0 & b+3 & 7 \end{array}\right]$$

a) need $a \neq 0$ (so x_2 is not free)
$b+3 \neq 0$, $b \neq -3$

b) $b+3 \neq 0$ and $a = 0$ (so x_2 is free)

c) $b+3=0$, $b=-3$

7. (6pts) Find the elementary matrix E so that $EA = B$.

$$A = \begin{bmatrix} 0 & -2 & 7 \\ 3 & 2 & -1 \\ -4 & 6 & 0 \end{bmatrix} \text{ swap} \qquad B = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -2 & 7 \\ -4 & 6 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. (12pts) Consider the vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ -3 \end{bmatrix}$.

a) Do they span $\mathbf{R}^4$?
b) Are they linearly independent?

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 0 & 3 & -2 \\ 1 & 1 & -3 \end{bmatrix} \begin{array}{l} \\ \cdot(-2) \\ \\ \cdot(-1) \end{array} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 3 & -2 \\ 0 & 1 & -2 \end{bmatrix} \begin{array}{l} \\ \\ \cdot 3 \\ + \end{array} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 7 \\ 0 & 0 & 5 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow \cdot\frac{1}{7} \\ \cdot(-5) \end{array} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ & & \end{bmatrix}$$

a) no, because rank $A = 3 <$ no. of rows

b) yes, because rank $A = 3 =$ no. of columns

9. (18pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If **a** and **b** are in Span{**u**, **v**}, then **a** −3**b** is in Span{**u**, **v**}.
b) If A is a 3×3 matrix with rank 2, then the columns of A are linearly dependent.
c) If 2×2 matrices A and B are invertible, then $A + B$ is invertible.

a) True. Let $\vec{a} = c_1\vec{u} + c_2\vec{v}$, $\vec{b} = d_1\vec{u} + d_2\vec{v}$

Then $\vec{a} - 3\vec{b} = c_1\vec{u} + c_2\vec{v} - 3(d_1\vec{u} + d_2\vec{v}) = (c_1 - 3d_1)\vec{u} + (c_2 - 3d_2)\vec{v}$

which is in span $\{\vec{u}, \vec{v}\}$

b) True. If rank $A = 2 <$ no of colums, then colums are lin. dependent

c) False: countexample: $A = I$ $(A^{-1} = A)$ $\quad$ $A + B = 0$, which
$B = -I$ $(B^{-1} = B)$ $\quad$ does not have an inverse,

Bonus. (10pts) Show: if vectors **u** and **v** are linearly independent, then vectors **u** + **v** and **u** −**v** are linearly independent.

Suppose $\vec{u}$ and $\vec{v}$ are linearly independent. Suppose

$c_1(\vec{u} + \vec{v}) + c_2(\vec{u} - \vec{v}) = \vec{0}$

$(c_1 + c_2)\vec{u} + (c_1 - c_2)\vec{v} = \vec{0}$

Since $\vec{u}$ and $\vec{v}$ are lin. independent

we must have $\quad c_1 + c_2 = 0$
$\quad c_1 - c_2 = 0$

$2c_1 = 0$, $c_1 = 0 \Rightarrow c_2 = 0$

We got that $c_1 = c_2 = 0$

so $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$

are lin. independent.