Matrix Theory — Lecture notes
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4.1 Subspaces

Definition. A nonempty set W of vectors in $\mathbf{R}^n$ is called a *subspace* of $\mathbf{R}^n$ if it has these two properties:

1. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W. (We say W is *closed under vector addition.*)
2. If $\mathbf{u}$ is in W, then $c\mathbf{u}$ is in W for any scalar c. (We say W is *closed under scalar multiplication.*)

Proposition. If W is a subspace of $\mathbf{R}^n$, then $\mathbf{0}$ is in W.

Proof.

Example. Are the following sets subspaces of $\mathbf{R}^3$?

$$W_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3 \mid 2x_1 + x_2 - 4x_3 = 1 \right\} \quad W_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3 \mid 2x_1 + x_2 - 4x_3 = 0 \right\}$$

Example. The one-element set $\{\mathbf{0}\}$ is a subspace of $\mathbf{R}^n$. The set $\mathbf{R}^n$ is a subspace of $\mathbf{R}^n$.

Example. Let W be the set of all vectors in $\mathbf{R}^2$ whose both components are integers. Is W a subspace of $\mathbf{R}^2$?

Example (Theorem 4.1). If $\mathcal{S}$ is any nonempty subset or $\mathbf{R}^n$, then $\operatorname{Span} \mathcal{S}$ is a subspace of $\mathbf{R}^n$. With this in mind, draw some subspaces of $\mathbf{R}^3$.

Definition (subspaces associated with a matrix). Let A be an $m \times n$ matrix.

— The *null space* of A is the solution set of $A\mathbf{x} = \mathbf{0}$, denoted by $\operatorname{Null} A$.
— The *column space* of A is the span of the columns of A, denoted by $\operatorname{Col} A$.
— The *row space* of A is the span of the rows of A, denoted by $\operatorname{Row} A$.

Theorem 4.2. Let A be an $m \times n$ matrix.

— The null space of A is a subspace of $\mathbf{R}^n$.
— The column space of A is a subspace of $\mathbf{R}^m$.
— The row space of A is a subspace of $\mathbf{R}^n$.

Proof.

Example. Write the generating sets for the null space, the column space and row space for the matrix below.

$$A = \begin{bmatrix} 1 & -4 & 7 & 1 & 3 \\ 3 & -10 & 27 & 2 & -1 \\ 1 & -3 & 10 & -1 & 5 \\ 2 & -8 & 14 & 4 & -3 \end{bmatrix}$$

Subspaces associated with a linear transformation. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with standard matrix A. Recall that the null space of T is all vectors x so that $T(\mathbf{x}) = \mathbf{0}$.

— The null space of T is the same as the null space of A, so it is a subspace of $\mathbf{R}^n$.
— The range of T is the column space of A, so it is a subspace of $\mathbf{R}^m$.

4.2 Basis and Dimension

Definition. Let V be a nonzero subspace of $\mathbf{R}^n$. A *basis* for V is a linearly independent generating set.

Example.

$\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbf{R}^n$

Any two nonparallel vectors in $\mathbf{R}^2$ are a basis for $\mathbf{R}^2$

Recall these theorems:

Theorem 1.7. $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}\} = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ if and only if $\mathbf{v}$ is in $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$.

Theorem 1.9. $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly dependent if and only if $\mathbf{u}_1 = \mathbf{0}$ or there is an $i \geq 2$ such that $\mathbf{u}_i$ is a linear combination of its predecessors.

Theorem 4.3 (Reduction Theorem). If $\mathcal{S}$ is a finite generating set for a nonzero subspace V or $\mathbf{R}^n$, then $\mathcal{S}$ can be reduced to a basis for V by removing vectors from $\mathcal{S}$.

Proof.

Example. Reduce the following generating set to a basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 11 \end{bmatrix} \right\}$

Theorem 4.4 (Extension Theorem). Let $\mathcal{S}$ be a linearly independent subset of a nonzero subspace V of $\mathbf{R}^n$. Then $\mathcal{S}$ can be extended to a basis for V by including additional vectors from V in $\mathcal{S}$. In particular, every nonzero subspace has a basis, which means every subspace is a span of finitely many vectors.

Proof.

Theorem 4.5. Let V be a subspace of $\mathbf{R}^n$. Then any two bases for V contain the same number of elements.

Proof.

Definition. The number of vectors in a basis for a nonzero subspace V or $\mathbf{R}^n$ is called the *dimension* of V, denoted $\dim V$. We define the dimension of the zero subspace to be 0.

Example. Is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ 3 \\ 0 \end{bmatrix} \right\}$ a basis for $\mathbf{R}^4$?

Theorem 4.6. Let V be a k-dimensional subspace of $\mathbf{R}^n$. Then

— Every linearly independent set of V contains at most k vectors.

— Every subset with more than k vectors is linearly independent.

Proof.

Theorem 4.7. Let V be a k-dimensional subspace of $\mathbf{R}^n$, and $\mathcal{S}$ a set with k vectors. Then

— If $\mathcal{S}$ is linearly independent, it is a basis for V.

— If $\mathcal{S}$ is a generating set for V, it is a basis for V.

Proof.

Example. Find a basis for the null space and column space of the matrix below.

$$A = \begin{bmatrix} -3 & -6 & 3 & -2 \\ -2 & -5 & 3 & 5 \\ 4 & 9 & -5 & 6 \end{bmatrix}$$

Example.

Is $\left\{ \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 21 \\ -17 \end{bmatrix} \right\}$ a basis for the subspace generated by $\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$?

Summary. To show a set $\mathcal{B}$ is a basis for a subspace V of $\mathbf{R}^n$:

1. Show $\mathcal{B}$ is contained in V.
2. Show $\mathcal{B}$ is linearly independent or that it generates V.
3. Compute $\dim V$ and check it equal the number of vectors in $\mathcal{B}$.

4.3 Dimensions of Subspaces Associated with a Matrix

Proposition. Let A be and $m \times n$ matrix. The following are dimensions of subspaces associated with the matrix A.

Subspace	Notation	Subspace of	Dimension
column space of A	$\operatorname{Col} A$	$\mathbf{R}^m$	$\operatorname{rank} A$
null space of A	$\operatorname{Null} A$	$\mathbf{R}^n$	$\operatorname{nullity} A = n - \operatorname{rank} A$
row space of A	$\operatorname{Row} A$	$\mathbf{R}^n$	$\operatorname{rank} A$

Proof.

Theorem 4.8. The nonzero rows in the reduced row echelon form of a matrix are the basis of the row space of the matrix.

Proof. We show that $\operatorname{Row} A = \operatorname{Row} R$.

Note. $\operatorname{rank} A = \operatorname{rank} A^T$

Theorem 4.9. If V and W are subspaces of $\mathbf{R}^n$ and V is contained in W, then $\dim V \leq \dim W$. If, additionally, $\dim V = \dim W$, then $V = W$.