Matrix Theory — Lecture notes
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3.1 Cofactor Expansion of Determinants

Example. Find a simple condition under which the system below has a solution for every $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Because the corresponding matrix A is 2×2, having a solution for every $\mathbf{b}$ automatically guarantees it is unique, because it boils down to rank $A = 2$, which is further equivalent to matrix A being invertible.

$$\begin{cases} ax_1 + bx_2 = & b_1 \\ cx_1 + dx_2 = & b_2 \end{cases}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Proposition. The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$.

Because the scalar $ad - bc$ *determines* whether A is invertible, it is called the *determinant* of A and denoted $\det A$ (or $|A|$, when the matrix is written out). For square matrices, the determinant is defined recursively, by "expansion along a row".

Example. Find the determinant of the matrix below by expanding it
a) along the first row b) along the second row.

$$\begin{vmatrix} 7 & 1 & 1 \\ 4 & -2 & 0 \\ -1 & 3 & -4 \end{vmatrix}$$

Definition. Let A be an $n \times n$ matrix, and let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained by removing the i-th row and j-th column of A. The *determinant* of A by expansion along row i is the number

$$\det A = a_{i1}(-1)^{i+1} \det A_{i1} + a_{i2}(-1)^{i+2} \det A_{i2} + \cdots + a_{ij}(-1)^{i+j} \det A_{ij} + \cdots + a_{in}(-1)^{i+n} \det A_{in}$$

If we set $c_{ij} = (-1)^{i+j} \det A_{ij}$, and call c_{ij} the (i,j)-*cofactor of* A, we can write this as

$$\det A = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in}$$

and call it the cofactor expansion of A along row i.

Theorem 3.1. For all $i = 1, \ldots, n$, the determinant of A by expansion along row i results in the same number.

Note. The determinant of a 2×2 matrix is essentially expansion along the first row, if we understand the determinant of a 1×1 matrix to be the single scalar it contains.

Example. Find the determinant of the matrix below.

$$\begin{vmatrix} 4 & 7 & 3 & 0 & 2 \\ 2 & -9 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 4 & -1 & 1 & 2 \\ 4 & -3 & 0 & 5 & 0 \end{vmatrix} =$$

Note.

To compute a	we have
2×2 determinant	2 multiplications, 1 addition
3×3 determinant	$3 \cdot 2$ multiplications, $3 \cdot 1 + (3-1) = 5 = 3! - 1$ additions
4×4 determinant	$4 \cdot 3 \cdot 2$ multiplications, $4(3! - 1) + 3 = 4! - 1$ additions
5×5 determinant	$5 \cdot 4 \cdot 3 \cdot 2$ multiplications, $5(4! - 1) + 4 = 5! - 1$ additions
$\vdots$	$\vdots$
$n \times n$ determinant	$n!$ multiplications, $n! - 1$ additions

For a smallish 20×20 determinant, this is about 4.87×10^{18} multiplications and additions. If each operation takes 10^{-9} seconds on a computer, this takes some 154 years — impractical! This is not the way to compute large determinants.

Definition. An $n \times n$ matrix is

— *upper triangular*, if all entries below the diagonal are 0
— *lower triangular*, if all entries above the diagonal are 0

Example. A is upper and B is lower triangular. Find $\det A$.

$$A = \begin{bmatrix} 7 & 1 & 1 & -3 \\ 0 & -2 & 0 & 5 \\ 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -1 & 2 & -4 & 0 \\ 0 & 1 & 7 & 5 \end{bmatrix}$$

Theorem 3.2. The determinant of an upper or lower triangular matrix is the product of the diagonal entries. In particular, $\det I_n = 1$.

Proposition. Let $\mathbf{a}, \mathbf{b}$ be vectors in $\mathbf{R}^2$, and $\mathbf{c}, \mathbf{d}, \mathbf{e}$ vectors in $\mathbf{R}^3$.

— The area of a parallelogram spanned by $\mathbf{a}, \mathbf{b}$ is $\left|\det \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix}\right|$.
— The volume of the parallelepiped spanned by $\mathbf{c}, \mathbf{d}, \mathbf{e}$ is $\left|\det \begin{bmatrix} \mathbf{c} & \mathbf{d} & \mathbf{e} \end{bmatrix}\right|$.

Example. Find the area of the triangle ABC, if $A = (1,3)$, $B = (3,0)$ and $C = (2,-1)$.

Note. Getting zero for the determinants above tells us $\mathbf{a}$ and $\mathbf{b}$ are parallel, and vectors $\mathbf{c}, \mathbf{d}$ and $\mathbf{e}$ are in the same plane. In other words, $\{\mathbf{a}, \mathbf{b}\}$ does not span $\mathbf{R}^2$, and $\{\mathbf{c}, \mathbf{d}, \mathbf{e}\}$ does not span $\mathbf{R}^3$, which we already knew, because the matrices involved are not invertible.

3.2 Properties of Determinants

Theorem 3.4. Let A and B $n \times n$ matrices. The following statements are true:

(a) A is invertible if and only if $\det A \neq 0$.
(b) $\det(AB) = \det A \cdot \det B$
(c) $\det A^T = \det A$
(d) If A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$

Proof. Statement b) is harder to prove, and we omit it. Assuming b), we can easily show a), d) and c).

Proposition. Let A be an invertible $n \times n$ matrix, and let C be the *cofactor matrix* of A: its (i, j)-entry is the (i, j)-cofactor of the matrix A. Then

$$A^{-1} = \frac{1}{\det A} C^T$$

In particular, if $ad - bc \neq 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note. Thanks to statement c), we can expand determinants by columns as well.

Theorem 3.3.
Let A be an $n \times n$ matrix.

If B is obtained from A by:	Then:
swapping two rows of A	$\det B = -\det A$
multiplying a row of A by k	$\det B = k \det A$
adding a multiple of a row to another row	$\det B = \det A$

Proof. Since $B = EA$, for some elementary matrix E, so $\det B = \det E \cdot \det A$, we just need to find determinants of elementary matrices.

Proposition. If a square matrix has two proportional rows, its determinant is 0.

Example. Use row operations to find the determinant of the matrix below, a much more efficient method than expansion by rows for large matrices.

$$\begin{vmatrix} 4 & 3 & 0 & 2 \\ 2 & 1 & -2 & 3 \\ 2 & -1 & 1 & 2 \\ 4 & 0 & 5 & 2 \end{vmatrix} =$$

Theorem 3.5 (Cramer's Rule). Let A be an invertible $n \times n$ matrix, $\mathbf{x}$ in $\mathbf{R}^n$ and let M_i be the matrix obtained from A by replacing column i of A by $\mathbf{b}$. Then the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ with components

$$x_i = \frac{\det M_i}{\det A} \text{ for } i = 1, \ldots, n.$$

Proof.

Example. Solve the system below with Cramer's rule.

$$\begin{cases} 2x_1 & -3x_2 & & = & 1 \\ x_1 & +2x_2 & +3x_3 & = & 4 \\ -4x_1 & +2x_2 & -5x_3 & = & 0 \end{cases}$$