Calculus 1 — Final Exam
MAT 250, Fall 2022 — D. Ivanšić

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Show all your work!

1. (16pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

$\lim_{x\to -3^+} f(x) = -\infty$

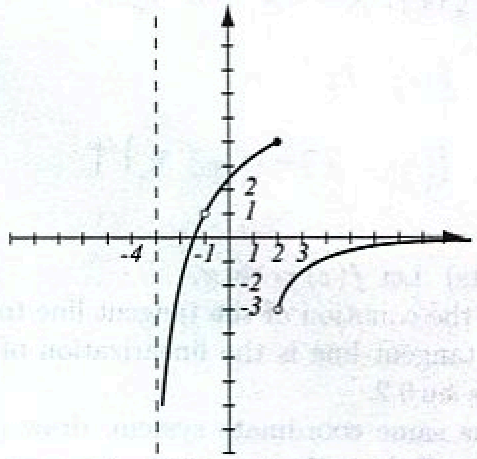
$\lim_{x\to 2^-} f(x) = 4$

$\lim_{x\to 2^+} f(x) = -3$

$\lim_{x\to 2} f(x) =$ does not exist, one sided limits are different

$\lim_{x\to -1} f(x) = 1$

$\lim_{x\to \infty} f(x) = 0$



List points where f is not continuous and explain why.

f is not cont. at $x=1$ b/c $f(1)$ is not defined

at $x=2$ b/c $\lim_{x\to 2} f(x)$ does not exist

Find the following limits algebraically. Do not use L'Hospital's rule.

2. (6pts) $\lim_{x\to 2^-} \frac{x+7}{3x-6} = \frac{9}{0^-} = 9\cdot\frac{1}{0^-} = 9\cdot(-\infty) = -\infty$

or: $x<2$, $3x<6$, $3x-6<0$

or: $x=1.99$, $3\cdot 1.99-6<0$

$-$ / $+$, 2

3. (5pts) $\lim_{x\to 2} \frac{x^2+2x-8}{x^2-9x+14} = \lim_{x\to 2} \frac{\cancel{(x-2)}(\cancel{x+4})}{\cancel{(x-2)}(\cancel{x-7})} = \lim_{x\to 2}\frac{x+4}{x-7} = \frac{6}{-5} = -\frac{6}{5}$

error canceling

4. (6pts) The equation $x^3 - 2x^2 + x = 5$ is given. Use the Intermediate Value Theorem to show it has a solution.

Let $f(x)=x^3-2x^2+x$ cont. on $\mathbb{R}$

$f(0)=0$

$f(3)=27-18+5=14$

Since $\underset{f(0)}{0}<5<\underset{f(3)}{14}$ by IVT

there is a c in $(0,3)$ such

that $f(c)=5$

5. (12pts) Let $f(x) = \sin x$.

a) Find the equation of the tangent line to the graph of f where $x = 0$.

b) The tangent line is the linearization of the function at $x = 0$. Use the linearization to estimate $\sin 0.2$.

c) In the same coordinate system, draw graphs of the function and the linearization and determine if the estimate overshoots or undershoots $\sin 0.2$.

a) $f'(x)=\cos x$

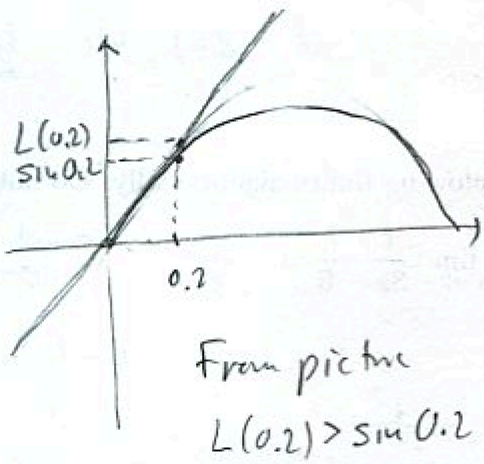
$f'(0)=1, \quad f(0)=0$

$y-0=1\cdot(x-0)$

$y=x$

b) $L(x)=x$

So $L(0.2)=0{,}2$

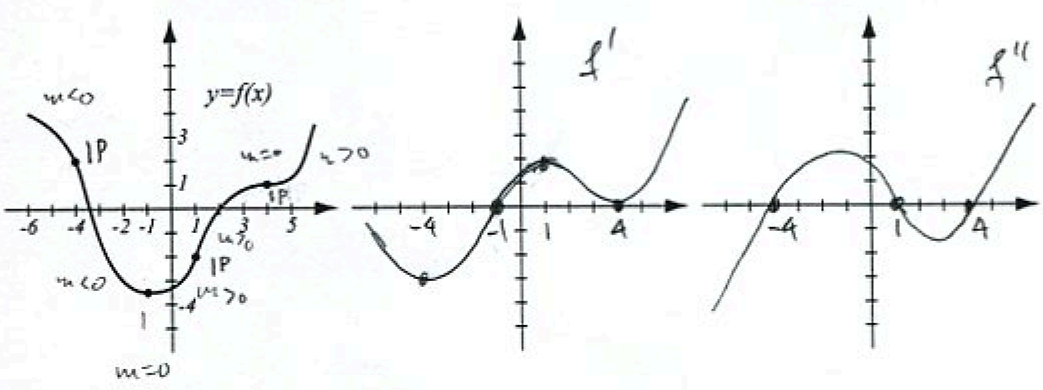


From picture

$L(0.2)>\sin 0.2$

overshoots

6. (12pts) The graph of f is given. Use it to draw the graphs of f' and f'' in the coordinate systems provided. Pay attention to increasingness, decreasingness and concavity of f. The relevant special points have been highlighted.



7. (26pts) Let $f(x) = \dfrac{4-x^2}{x^2+9}$. Draw an accurate graph of f by following the guidelines.

a) Find the intervals of increase and decrease, and local extremes.

b) Find the intervals of concavity and points of inflection.

c) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

d) Use information from a)–c) to sketch the graph.

$$f'(x) = \frac{-2x(x^2+9)-(4-x^2)\cdot 2x}{(x^2+9)^2} = \frac{\cancel{-2x^3}-18x-8x+\cancel{2x^3}}{(x^2+9)^2}$$

$$= \frac{-26x}{(x^2+9)^2}$$

$$f''(x) = -26\cdot\frac{1\cdot(x^2+9)^2 - x\cdot 2(x^2+9)\cdot 2x}{(x^2+9)^4} \quad 9-3x^2$$

$$= -26\,\frac{\cancel{(x^2+9)}(x^2+9-4x^2)}{\cancel{(x^2+9)^4}\,{}^{3}} = -26(-3x^2+9)$$

$$= \frac{78(x^2-3)}{(x^2+9)^3}$$

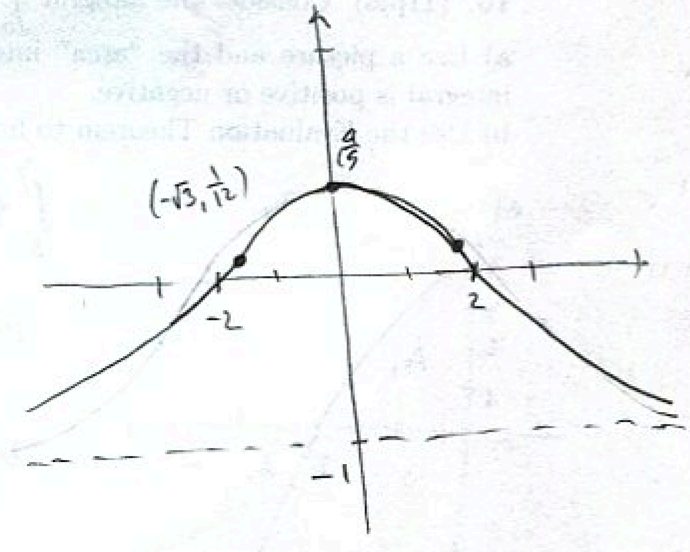
c) $\lim\limits_{\substack{x\to\infty\\(-\infty)}} \dfrac{4-x^2}{x^2+9} = \lim\limits_{\substack{x\to\infty\\(-\infty)}} \dfrac{\cancel{x^2}\left(\frac{4}{x^2}-1\right)}{\cancel{x^2}\left(1+\frac{9}{x^2}\right)}$ $\quad \frac{4}{x^2}\to 0,\ \frac{9}{x^2}\to 0$

$= \dfrac{0-1}{1+0} = -1$

Same lim for $x\to-\infty$.

d)

x	$f(x)$
0	$\frac{4}{9}$
$-\sqrt{3}$, $\sqrt{3}$	$\frac{4-3}{3+9} = \frac{1}{12}$



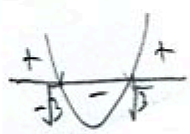
a) $x^2+9=0$ never $\qquad x=0$

$(x^2+9)^2>0$ so sign of f' depends on x

		0	
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$	loc. max	$\searrow$

b) $x^2+9=0$ never $\qquad x^2-3=0$, $x^2=3$, $x=\pm\sqrt{3}$

Since $(x^2+9)^3>0$, f'' has sign of x^2-3



		$-\sqrt{3}$		$\sqrt{3}$	
f''	$+$	0	$-$	0	$+$
f	CU	IP	CD	IP	CU

8. (10pts) Let $f(x) = \dfrac{x^2}{e^x}$. Find the absolute minimum and maximum values of f on the interval $[-1, 3]$.

$f'(x) = \dfrac{2xe^x - x^2e^x}{(e^x)^2} = \dfrac{e^x(2x-x^2)}{(e^x)^2}$

$= \dfrac{2x-x^2}{e^x}$

$f'(x) = 0 \quad 2x - x^2 = 0$

$x(2-x) = 0$

$x = 0, 2$

$f'(x)$ not def.

$e^x = 0$

never

x	$\frac{x^2}{e^x}$	
0	0	abs min
2	$\frac{4}{e^2} \approx \frac{4}{9}$	
-1	$\frac{1}{e^{-1}} = e \approx 3$	abs. max
3	$\frac{9}{e^3} \approx \frac{9}{27} = \frac{1}{3}$	

9. (6pts) Find $f(x)$ if $f'(x) = \sqrt[4]{x^3} + \dfrac{1}{x}$ and $f(1) = 3$.

$f'(x) = x^{\frac{3}{4}} + \frac{1}{x}$

$f(x) = \frac{4}{7}x^{\frac{7}{4}} + \ln|x| + C$

$3 = f(1) = \frac{4}{7} \cdot 1^{\frac{7}{4}} + \ln 1 + C$

$3 = \frac{4}{7} + C$

$C = 3 - \frac{4}{7} = \frac{17}{7}$

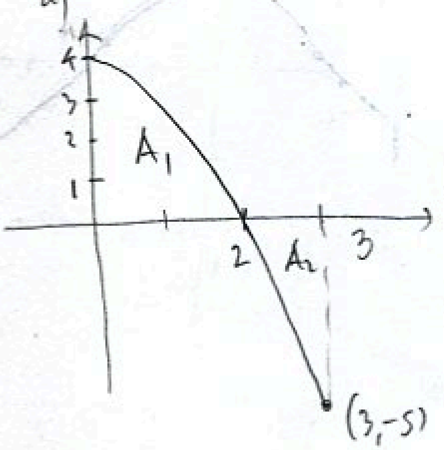
$f(x) = \frac{4}{7}x^{\frac{7}{4}} + \ln|x| + \frac{17}{7}$

10. (11pts) Consider the integral $\displaystyle\int_0^3 4 - x^2\, dx$.

a) Use a picture and the "area" interpretation of the integral to determine whether this integral is positive or negative.

b) Use the Evaluation Theorem to find the integral and verify your conclusion from a).

a)



$\displaystyle\int_0^3 4 - x^2\, dx = A_1 - A_2$

Since visually it appears that $A_1 > A_2$,

$A_1 - A_2 > 0$

integral positive

b) $\displaystyle\int_0^3 4 - x^2\, dx = \left(4x - \frac{x^3}{3}\right)\Big|_0^3 = 4 \cdot 3 - \frac{3^3}{3} - 0$

$= 12 - 9 = 3$, it is positive

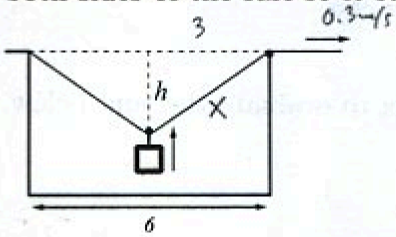
11. (12pts) Toxic sludge is being deposited into a collection pool at rate $3\sqrt{t}+t$ cubic meters per hour.
a) Use the Net Change Theorem to find how much sludge was added from $t=0$ to $t=4$ hours.
b) If at time $t=0$ there were 12 cubic meters of sludge in the pool, how much is there at $t=4$ hours?

a) $\Delta V =$ amount added $= \int_0^4$ rate of adding $dt = \int_0^4 3\sqrt{t}+t\,dt = \left(3\cdot\frac{2}{3}t^{\frac{3}{2}}+\frac{t^2}{2}\right)\Big|_0^4$

$= 2\cdot 4^{\frac{3}{2}} + \frac{4^2}{2} - 0 = 16+8 = 24$

b) $V(4) = V(0) + \Delta V = 12 + 24 = 36 \text{ m}^3$

12. (12pts) A safe is raised using the pulley system shown, with rope pulled at right at speed 0.3 meters per second, while it is anchored at left. How fast is the distance to the top h changing when the middle pulley is 1 meter from the top? The rope shortens evenly on both sides of the safe so it stays in the center.



Need h' when $h=1$
Know $x' = -0.15$
(since shortening is split over 2 pieces of rope)

$h^2+3^2=x^2 \quad \Big| \frac{d}{dt}$

$2hh'+0=2xx'$

$hh' = xx'$

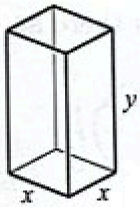
$h' = \frac{xx'}{h}$

When $h=1$:
$1^2+3^2=x^2$
$x^2=10$
$x=\sqrt{10}$

$h' = \frac{\sqrt{10}\cdot(-0.15)}{1}$

$= -0.15\sqrt{10}$ m/s

13. (16pts) A box with a square base uses 98 square inches of material (that is its surface area, including top and bottom). Find the dimensions x, y of the box that give the maximal possible volume of the box. Don't forget to check it is a maximum.



$V = x^2 y = \cancel{x^2}\left(\frac{49-x^2}{2\cancel{x}}\right) = \frac{1}{2}x(49-x^2) = \frac{1}{2}(49x - x^3) = V(x)$

Surface area $= 2x^2 + 4xy = 98$

Job maximize $V(x)$ on $(0,7]$

$x^2 + 2xy = 49$

$2xy = 49 - x^2$

$y = \dfrac{49-x^2}{2x}$

Must have $y \geq 0$

$49 - x^2 \geq 0$

$x^2 \leq 49$

$x \leq 7$

$V'(x) = \frac{1}{2}(49 - 3x^2)$

$V'(x) = 0$ when $49 - 3x^2 = 0$

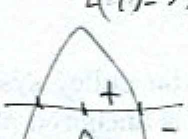
$3x^2 = 49$

$x^2 = \frac{49}{3}$

$x = \pm\frac{7}{\sqrt{3}} = \frac{7}{\sqrt{3}}$ since $x > 0$

Check it is a max:

$\frac{1}{2}(49 - 3x^2)$



	0		$\frac{7}{\sqrt{3}}$	
$V'(x)$		$+$		$-$
$V(x)$		$\nearrow$	loc. max (and abs)	$\searrow$

Bonus. (10pts) Recall that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Use this formula to evaluate the sum below. (Note that it does not start with 1, how do you handle this?)

$$\sum_{i=4}^{n}(7+2i) = \sum_{i=1}^{n}(7+2i) - \sum_{i=1}^{3}(7+2i)$$

$$= \sum_{i=1}^{n} 7 + 2\sum_{i=1}^{n} i - (7 + 2\cdot1 + 7 + 2\cdot2 + 7 + 2\cdot3)$$

$$= 7n + 2\cdot\frac{n(n+1)}{2} - 33$$

$$= 7n + n^2 + n - 33 = n^2 + 8n - 33 \quad \text{for } n \geq 4$$