Calculus 1 — Exam 5
MAT 250, Fall 2022 — D. Ivanšić

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Show all your work!

Find the following antiderivatives or definite integrals.

1. (3pts) $\int \sqrt[5]{x^2}\,dx =$ $\int x^{\frac{2}{5}}\,dx = \frac{x^{\frac{7}{5}}}{\frac{7}{5}} = \frac{5}{7}x^{\frac{7}{5}} + C$

2. (3pts) $\int \cos\left(3x + \frac{\pi}{2}\right)\,dx =$ $\frac{\sin\left(3x+\frac{\pi}{2}\right)}{3} + C$

3. (6pts) $\int \sqrt{t}(t^2 - 3t)\,dt =$ $\int t^{1/2}(t^2-3t)\,dt = \int t^{\frac{5}{2}} - 3t^{3/2}\,dt = \frac{t^{\frac{7}{2}}}{\frac{7}{2}} - 3\cdot\frac{t^{\frac{5}{2}}}{\frac{5}{2}}$

$= \frac{2}{7}t^{\frac{7}{2}} - \frac{6}{5}t^{\frac{5}{2}} + C$

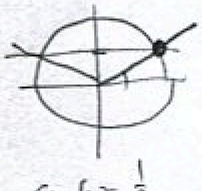
4. (4pts) $\int_1^e \frac{1}{x}\,dx =$ $\ln|x|\Big|_1^e = \ln e - \ln 1 = 1 - 0 = 1$

5. (7pts) $\int_0^{\frac{\pi}{6}} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta =$ $\int_0^{\pi/6} \frac{1}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/6} \sec^2\theta + 1\,d\theta = (\tan\theta + \theta)\Big|_0^{\frac{\pi}{6}}$

$= \tan\frac{\pi}{6} - \tan 0 + \frac{\pi}{6} - 0 = \frac{1}{\sqrt{3}} + \frac{\pi}{6}$

6. (6pts) Find $f(x)$ if $f'(x) = \frac{4}{\sqrt{1-x^2}}$ and $f\left(\frac{1}{2}\right) = 3$.

$f(x) = 4\arcsin x + C$



$3 = f(\tfrac{1}{2}) = 4\arcsin\tfrac{1}{2} + C$ $\qquad f(x) = 4\arcsin x + 3 - \frac{2\pi}{3}$

$3 = 4\cdot\frac{\pi}{6} + C$

$C = 3 - \frac{2\pi}{3}$

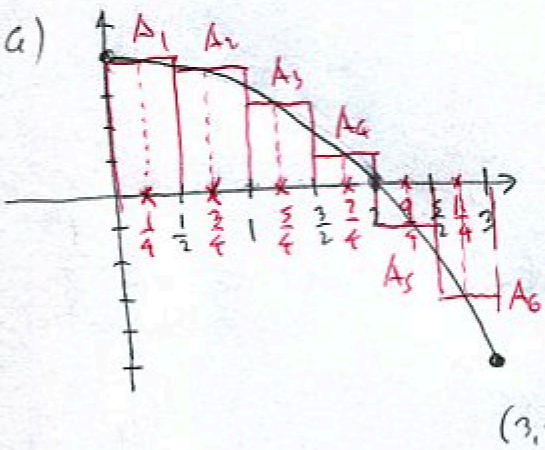
$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

arcsin

7. (15pts) The function $f(x) = 4 - x^2$ is given on the interval $[0, 3]$.
a) Write the Riemann sum M_6 for this function with six subintervals, taking sample points to be midpoints. Do not evaluate the expression.
b) Illustrate with a diagram, where appropriate rectangles are clearly visible. What does M_6 represent?

a)

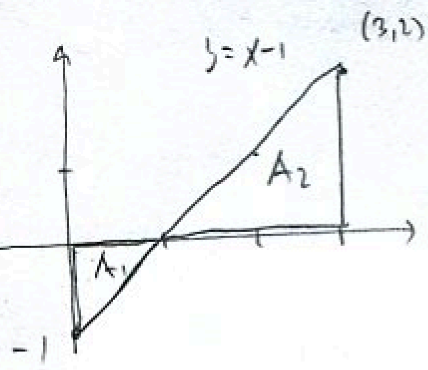


a) $M_6 = \frac{1}{2}\Big(4-(\frac{1}{4})^2 + 4-(\frac{3}{4})^2 + 4-(\frac{5}{4})^2$
$+ 4-(\frac{7}{4})^2 + 4-(\frac{9}{4})^2 + 4-(\frac{11}{4})^2\Big)$

b) If $A_1, \dots, A_6$ are areas of rectangles,
then $M_6 = A_1 + A_2 + A_3 + A_4 - A_5 - A_6$

8. (13pts) Find $\int_0^3 x - 1\,dx$ in two ways (they'd better give you the same answer!):
a) Using the "area" interpretation of the integral. Draw a picture.
b) Using the Evaluation Theorem.

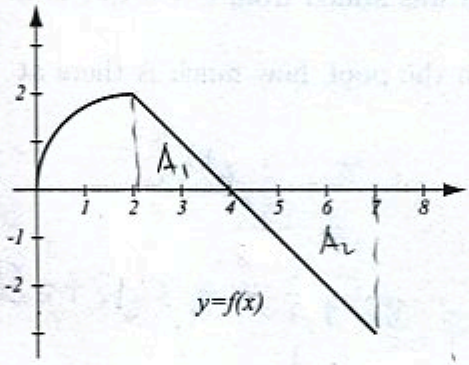
a)



a) $\int_0^3 x-1\,dx = -A_1 + A_2 = -\frac{1}{2}1.1 + \frac{1}{2}2.2$
$= -\frac{1}{2} + 2 = \frac{3}{2}$

b) $\int_0^3 x-1\,dx = \left(\frac{x^2}{2} - x\right)\Big|_0^3 = \frac{1}{2}(3^2-0^2) - (3-0)$
$= \frac{9}{2} - 3 = \frac{3}{2}$

9. (10pts) The graph of a function f, consisting of lines and parts of circles, is shown. Evaluate the integrals.

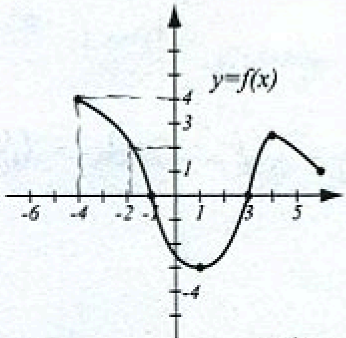


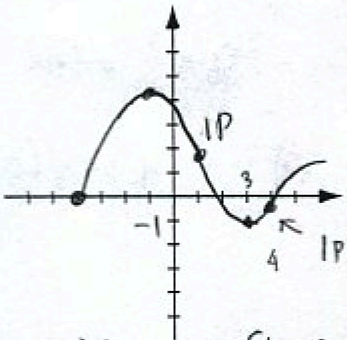
$\int_0^2 f(x)\,dx = \frac{1}{4}\cdot\pi\cdot 2^2 = \pi$

$\int_2^7 f(x)\,dx = A_1 - A_2 = \frac{1}{2}2\cdot2 - \frac{1}{2}3\cdot3 = \frac{4-9}{2} = -\frac{5}{2}$

$\int_0^7 f(x)\,dx = \int_0^2 + \int_2^7 = \pi - \frac{5}{2}$

10. (8pts) Pictured is the graph of $f(x)$. Sketch the graph of the antiderivative $F(x)$ of $f(x)$, if it is known that $F(-4) = 0$. Relevant points have been highlighted.





$F^{\text{incr}}_{\text{decr}} \Leftrightarrow f^{>0}_{<0}$ $\qquad F^{CU}_{CD} \Leftrightarrow f^{\text{incr}}_{\text{decr}}$

11. (7pts) Use the inequality $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$, where $m \le f(x) \le M$ on $[a,b]$, to give an estimate of $\int_{-4}^{-2} f(x)\,dx$, where f is the function pictured in the preceding problem.

On interval $[-4,-2]$ we see from graph

$2 \le f(x) \le 4$

$2(4-2) \le \int_{-4}^{-2} f(x)dx \le 4(4-2)$

$4 \le \int_{-4} f(x)dx \le 8$

12. (6pts) Write using sigma notation:

$\frac{2}{x^4} + \frac{3}{x^6} + \frac{4}{x^8} + \cdots + \frac{9}{x^{18}} = \sum_{i=2}^{9} \frac{i}{x^{2i}}$

13. (12pts) Toxic sludge is being deposited into a collection pool at rate $6\sqrt{t}+3t$ cubic meters per hour.
a) Use the Net Change Theorem to find how much sludge was added from $t=0$ to $t=4$ hours.
b) If at time $t=0$ there were 12 cubic meters of sludge in the pool, how much is there at $t=4$ hours?

a) $\Delta V = \int_0^4 6\sqrt{t} + 3t \, dt = \left(6\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + 3\cdot\frac{t^2}{2}\right)\Big|_0^4 = \left(4t^{3/2} + \frac{3}{2}t^2\right)\Big|_0^4$

$= 4\cdot 4^{3/2} + \frac{3}{2}\cdot 4^2 - 0 = 32 + \frac{3}{2}\cdot 16 = 32+24 = 56$

$(\sqrt{4})^3 = 8$

$56\,m^3$ added during times $t=0$ to $t=4$

b) $V(4) = V(0) + \Delta V = 12+56 = 68\ m^3$

Bonus. (10pts) Recall that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Use this formula to evaluate the sum below. (Note that it does not start with 1, how do you handle this?)

$\sum_{i=4}^{n}(7+2i) = \sum_{i=1}^{n}(7+2i) - (7+2\cdot1 + 7+2\cdot2 + 7+2\cdot3)$

first three terms

$= \sum_{i=1}^{n} 7 + 2\sum_{i=1}^{n} i - (21+12) = 7n + 2\frac{n(n+1)}{2} - 33$

$= n^2+n+7n-33 = n^2+8n-33$