Calculus 1 — Exam 4
MAT 250, Fall 2022 — D. Ivanšić

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1. (30pts) Let $f(x) = \dfrac{x^2}{x^2+9}$. Draw an accurate graph of f by following the guidelines.

a) Find the intervals of increase and decrease, and local extremes.
b) Find the intervals of concavity and points of inflection.
c) Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
d) Use information from a)–c) to sketch the graph.

$$f'(x) = \frac{2x(x^2+9) - x^2 \cdot 2x}{(x^2+9)^2} = \frac{2x^3+18x-2x^3}{(x^2+9)^2} = \frac{18x}{(x^2+9)^2}$$

a) $f'(x)=0$ at $x=0$ $\qquad$ $f'(x)$ undefined never

Since $(x^2+9)^2>0$, has sign of x

		0	
f'	$-$	0	$+$
f	$\searrow$	loc min	$\nearrow$

$$f''(x) = 18\,\frac{1\cdot(x^2+9)^2 - x\cdot 2(x^2+9)\cdot 2x}{(x^2+9)^4} = 18\,\frac{\cancel{(x^2+9)}(x^2+9-4x^2)}{(x^2+9)^{\cancel{4}\,3}} = 18\,\frac{9-3x^2}{(x^2+9)^3} = \frac{54(3-x^2)}{(x^2+9)^3}$$

b) $f''(x)=0$ $\qquad$ f'' always defined

$3-x^2=0$ $\qquad$ Since $(x^2+9)^3>0$

$x^2=3$ $\qquad$ f'' has sign of $3-x^2$

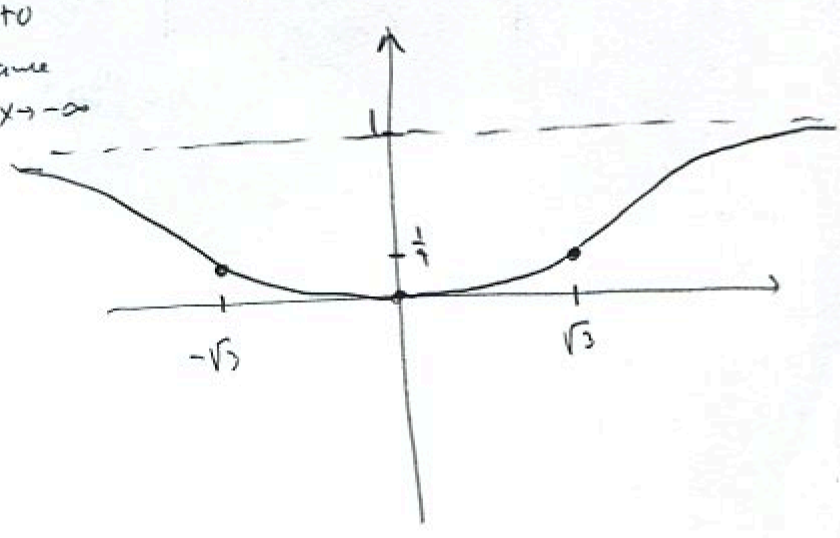
$x=\pm\sqrt{3}$ $\qquad$ $y=3-x^2$

		$-\sqrt{3}$		$\sqrt{3}$	
f''	$-$	0	$+$	0	$-$
f	CD	IP	CU	IP	CD

c) $\lim\limits_{\substack{x\to\infty \\ \to-\infty}} \dfrac{x^2}{x^2+9} = \lim\limits_{x\to\infty} \dfrac{x^2}{x^2(1+\frac{9}{x^2})} = \dfrac{1}{1+0} = 1$

same for $x\to-\infty$

x	$\frac{x^2}{x^2+9}$
0	0
$-\sqrt{3}$	$\frac{3}{3+9} = \frac{1}{4}$
$\sqrt{3}$	$\frac{3}{3+9} = \frac{1}{9}$

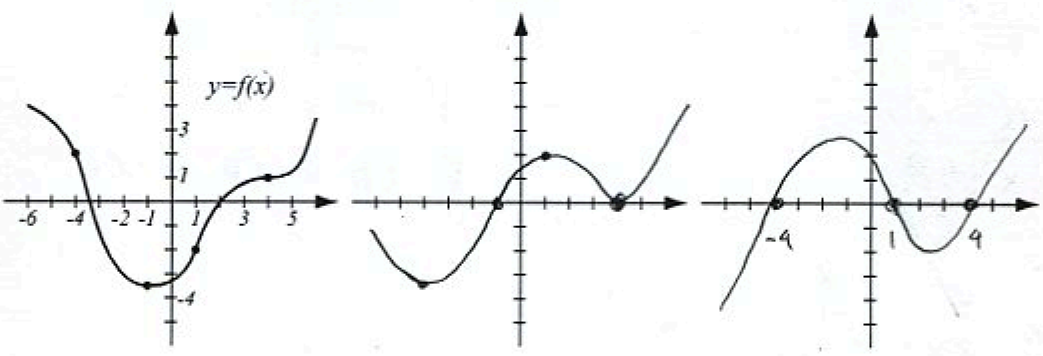


2. (16pts) Let $f(x) = x^2e^x$. Find the absolute minimum and maximum values of f on the interval $[-4, 1]$.

$$f'(x) = 2xe^x + x^2e^x$$
$$= e^x(x^2+2x)$$
$$e^x\,x(x+2) = 0$$
$$>0 \qquad x = 0 \text{ or } -2$$

x	x^2e^x	
0	0	abs min
-2	$4e^{-2} = \frac{4}{e^2}$	< 1
-4	$16e^{-4} = \frac{16}{e^4}$	< 1
1	e	abs max

3. (14pts) The graph of f is given. Use it to draw the graphs of f' and f'' in the coordinate systems provided. Pay attention to increasingness, decreasingness and concavity of f. The relevant special points have been highlighted.



$y = f'(x)$ $\qquad$ $y = f''(x)$

4. (~~16~~ 18pts) Consider $f(x) = \sin^2 x - \cos x$ on the interval $[0, 2\pi]$.
a) Verify that the function satisfies the assumptions of the Mean Value Theorem.
b) Find all numbers c that satisfy the conclusion of the Mean Value Theorem.

a) f is cont on $\mathbb{R}$, so on $[0, 2\pi]$

f is diff. on $\mathbb{R}$, so on $(0, 2\pi)$

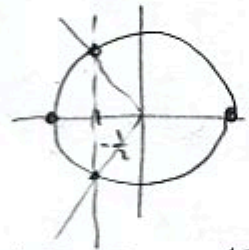
b) $\dfrac{f(2\pi) - f(0)}{2\pi - 0} = \dfrac{(0-1)-(0-1)}{2\pi} = 0$

$f'(x) = 2\sin x \cos x + \sin x$

$2\sin x \cos x + \sin x = 0$

$\sin x\,(2\cos x + 1) = 0$

$\sin x = 0$ or $\cos x = -\frac{1}{2}$

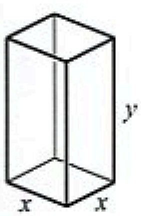


$x = 0, \pi, 2\pi$ $\qquad$ $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

solutions that are in $(0, 2\pi)$:

$\pi, \frac{2\pi}{3}, \frac{4\pi}{3}$

5. (22pts) A box with a square base uses 50 square inches of material (that is its surface area, including top and bottom). Find the dimensions of the box that give the maximal possible volume of the box.



$A = 2x^2 + 4xy = 50$

$x^2 + 2xy = 25$

$2xy = 25 - x^2$

$y = \dfrac{25 - x^2}{2x}$

must have $25 - x^2 \geq 0$

$x^2 \leq 25$

$x \leq 5$

$V = x^2 y = x^2 \cdot \dfrac{25 - x^2}{2x} = \frac{1}{2}x(25 - x^2)$

$= \frac{1}{2}(-x^3 + 25x)$

Job: maximize $V(x)$ on $(0, 5]$

$V'(x) = \frac{1}{2}(-3x^2 + 25)$

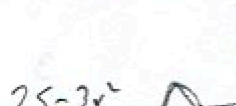
$V'(x) = 0 \quad 25 - 3x^2 = 0$

$3x^2 = 25$

$x^2 = \frac{25}{3}$

$x = \pm \frac{5}{\sqrt{3}}$

	0		$\frac{5}{\sqrt{3}}$	
V'		$+$	0	$-$
V		$\nearrow$	abs max	$\searrow$



$y = \dfrac{25 - \left(\frac{5}{\sqrt{3}}\right)^2}{2 \cdot \frac{5}{\sqrt{3}}} = \dfrac{25 - \frac{25}{3}}{\frac{10}{\sqrt{3}}} = \frac{2}{3} 25 \cdot \frac{\sqrt{3}}{10}$

$= \frac{5}{3}\sqrt{3} = \frac{5}{\sqrt{3}}$

V has abs. max at $x = \frac{5}{\sqrt{3}}$

Bonus. (10pts) Let $f(x)=\frac{5}{9}x^{\frac{9}{5}}-\frac{15}{4}x^{\frac{4}{5}}$. Note domain is all real numbers.
a) Find the intervals of increase and decrease, and local extremes.
b) Find the intervals of concavity and points of inflection.
c) Use information from a) and b) to sketch the graph.

$$f'(x)=\frac{5}{9}\cdot\frac{9}{5}x^{\frac{4}{5}}-\frac{15}{4}\cdot\frac{4}{5}x^{-\frac{1}{5}}$$
$$=x^{\frac{4}{5}}-3x^{-\frac{1}{5}}=x^{-\frac{1}{5}}(x-3)=\frac{x-3}{\sqrt[5]{x}}$$
$$f''(x)=\frac{4}{5}x^{-\frac{1}{5}}-3\left(-\frac{1}{5}\right)x^{-\frac{6}{5}}=x^{-\frac{6}{5}}\left(\frac{4}{5}x+\frac{3}{5}\right)=\frac{4x+3}{5\sqrt[5]{x^6}}$$

$f'(x)=0$
$x-3=0$
$x=3$

$f'(x)$ DNE
$\sqrt[5]{x}=0$
$x=0$

$f''(x)=0$
$4x+3=0$
$x=-\frac{3}{4}$

$f''(x)$ not def
$\sqrt[5]{x^6}=0$
$x=0$

$\sqrt[5]{x^6}=(\sqrt[5]{x})^6\geq 0$

		0		3	
$\sqrt[5]{x}$	$-$	0	$+$		$+$
$x-3$	$-$		$-$	0	$+$
f'	$+$	ND	$-$	0	$+$
f	$\nearrow$	loc max (vert. tan)	$\searrow$	loc. min	$\nearrow$

		$-\frac{3}{4}$	
$f''(x)$	$-$	0	$+$
f	CD	IP	CU

x	$f(x)=x^{\frac{4}{5}}\left(\frac{5}{9}x-\frac{15}{4}\right)$
$-\frac{3}{4}$	$\left(-\frac{3}{4}\right)^{\frac{4}{5}}\left(-\frac{5}{9}\left(\frac{3}{4}\right)-\frac{15}{4}\right)=\left(\frac{3}{4}\right)^{\frac{4}{5}}\frac{5-45}{12}=-\frac{10}{3}\left(\frac{3}{4}\right)^{\frac{4}{5}}$
0	0 $\quad \frac{5}{12}-\frac{15}{4}$
3	$3^{\frac{4}{5}}\left(\frac{5}{3}-\frac{15}{4}\right)=3^{\frac{4}{5}}\frac{-25}{12}<0$