Calculus 1 — Exam 3
MAT 250, Fall 2022 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (5pts) $\frac{d}{dx} e^{x^2-x+3} =$ $e^{x^2-x+3} \cdot (2x-1)$

2. (6pts) $\frac{d}{dx}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) e^x =$ $\left(\frac{1}{2\sqrt{x}} + \frac{1}{2} x^{-\frac{3}{2}}\right) e^x + \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) e^x$

$x^{-\frac{1}{2}}$

$= e^x\left(\sqrt{x} - \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x^3}}\right)$

3. (6pts) $\frac{d}{dt}\frac{\arctan t}{t^2} =$ $\frac{\frac{1}{1+t^2} t^2 - \arctan t \cdot 2t}{(t^2)^2} = \frac{\frac{t^2}{1+t^2} - 2t\arctan t}{t^4} \cdot \frac{1+t^2}{1+t^2}$

$= \frac{t^2 - 2t(1+t^2)\arctan t}{t^4(1+t^2)} = \frac{\cancel{t}(t - 2(1+t^2)\arctan t)}{{}^3\cancel{t^4}(1+t^2)} = \frac{t - 2(1+t^2)\arctan t}{t^3(1+t^2)}$

4. (7pts) $\frac{d}{dx} \ln\left(\frac{x+2}{x-2}\right)^3$ $= \frac{d}{dx} 3\ln\left(\frac{x+2}{x-2}\right) = 3\frac{d}{dx}\left(\ln(x+2) - \ln(x-2)\right)$

$= 3\left(\frac{1}{x+2} - \frac{1}{x-2}\right) = 3\,\frac{x-2-(x+2)}{(x+2)(x-2)} = \frac{-12}{(x+2)(x-2)}$

5. (7pts) $\frac{d}{d\theta}\ln(\sec\theta + \tan\theta) =$ $\frac{1}{\sec\theta + \tan\theta} \cdot \left(\sec\theta\tan\theta + \sec^2\theta\right)$

$= \frac{\sec\theta\cancel{(\tan\theta + \sec\theta)}}{\cancel{\sec\theta + \tan\theta}} = \sec\theta$

6. (9pts) Use logarithmic differentiation to find the derivative of $y = (\sin x)^{\cos x}$.

$y = (\sin x)^{\cos x}$

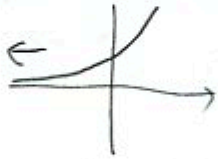
$\frac{d}{dx}$ $\ln y = \ln(\sin x)^{\cos x} = \cos x \ln \sin x$

$\frac{1}{y} \cdot y' = -\sin x \ln \sin x + \cos x \cdot \frac{1}{\sin x} \cdot \cos x \quad | \cdot y$

$y' = (\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x} - \sin x \ln \sin x\right)$

Find the limits algebraically. Graphs of basic functions will help, as will L'Hospital's rule, where appropriate.

7. (2pts) $\lim_{x\to-\infty} e^{3x} =$ $e^{-\infty} = 0$



8. (7pts) $\lim_{x\to\infty} \arctan\left(\frac{x^2+5x+1}{x+7}\right) =$ $\arctan \underset{\to 0}{\lim_{x\to\infty}} \left(\frac{x^2+5x+1}{x+7}\right)$

$$= \arctan \lim_{x\to\infty} \frac{\overset{x\to\infty}{x^{\cancel{2}}}\left(1+\underset{\to 0}{\frac{5}{x}}+\underset{\to 0}{\frac{1}{x^2}}\right)}{\cancel{x}\left(1+\underset{\to 0}{\frac{7}{x}}\right)} = \arctan\left(\infty\cdot\frac{1+0+0}{1+0}\right)$$

$$= \arctan\infty = \frac{\pi}{2}$$

9. (6pts) $\lim_{x\to\infty} \frac{\overset{\to\infty}{x^2}}{\underset{\to\infty}{2^x}} \overset{L'H}{=}$ $\lim_{x\to\infty}\frac{\overset{\to\infty}{2x}}{\underset{\to\ln 2\cdot\infty\,=\,\infty}{\ln 2\cdot 2^x}} \overset{L'H}{=} \lim_{x\to\infty}\frac{2}{(\ln 2)^2\cdot 2^x} = \frac{2}{(\ln 2)^2\cdot\infty} = \frac{2}{\infty} = 0$

10. (9pts) $\lim_{x\to 0} \frac{\overset{\to 0-0=0}{\tan x - x}}{\underset{\to 0}{x^3}} \overset{L'H}{=}$ $\lim_{x\to 0}\frac{\overset{\to 1-1=0}{\sec^2 x - 1}}{\underset{\to 0}{3x^2}} \overset{L'H}{=} \lim_{x\to 0}\frac{2\overset{\to 1}{\sec x}\,\overset{\to 0}{\tan x}}{\underset{\to 0}{6x}} = \lim_{x\to 0}\frac{1}{3}\sec x\cdot\lim_{x\to 0}\frac{\overset{\to 0}{\tan x}}{\underset{\to 0}{x}}$

$$= \frac{1}{3}\cdot 1\cdot\frac{\overset{\to 1}{\sec^2 x}}{1} = \frac{1}{3}$$

11. (8pts) $\lim_{x\to 0+} x^x =$ $e^0 = 1$

$\ln x^x = x\ln x$

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0}\frac{\overset{\to-\infty}{\ln x}}{\underset{\to\infty}{\frac{1}{x}}} \overset{L'H}{=} \lim_{x\to 0+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+} -\frac{x^2}{x} = \lim_{x\to\infty} -x = 0$$

12. (12pts) Let $f(x) = \ln x$.
a) Write the linearization of $f(x)$ at $a = 1$.
b) Use the linearization to estimate $\ln 1.2$.
c) In the same coordinate system, draw rough graphs of the function and the linearization and determine if the estimate overshoots or undershoots $\ln 1.2$.

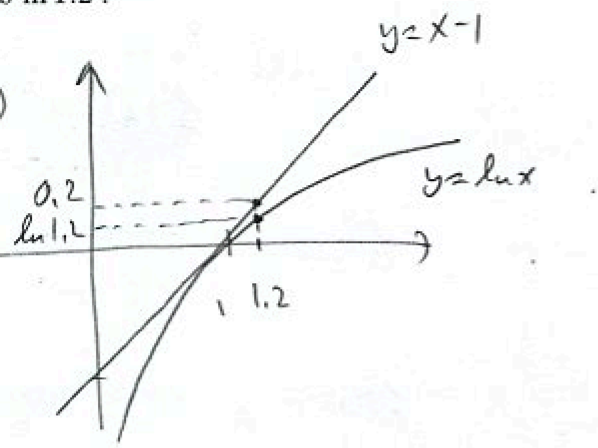
a) $f(x) = \ln x \qquad f(1) = 0$

$f'(x) = \frac{1}{x} \qquad f'(1) = 1$

$L(x) = 0 + 1 \cdot (x-1) = x - 1$

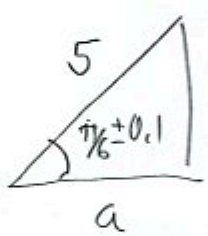
b) $\ln 1.2 \approx L(1.2) = 1.2 - 1 = 0.2$

c)



From picture: $0.2 > \ln 1.2$
overshoots

13. (9pts) In a right triangle, the hypothenuse is known to be 5 inches. One of the angles is measured to be $\frac{\pi}{6}$ radians, with maximum error 0.1 radians. Use differentials to estimate the maximum possible error and relative error when computing the length of the side adjacent to the angle.



$\frac{a}{5} = \cos\theta$

$a = 5\cos\theta$

$da = -5\sin\theta \, d\theta$

$da = -5\sin\frac{\pi}{6} \cdot 0.1 = -\cancel{5} \cdot \frac{1}{2} \cdot \frac{1}{\cancel{10}_{2}} = -\frac{1}{4}$ in

relative error $= \frac{da}{a} = \frac{-\frac{1}{4}}{5 \cdot \frac{\sqrt{3}}{2}} = -\frac{1}{\cancel{4}_{2}} \cdot \frac{\cancel{2}}{5\sqrt{3}} = -\frac{1}{10\sqrt{3}}$

$5\cos\frac{\pi}{6}$

14. (7pts) Let $f(x) = x^3 - x$. Use the theorem on derivatives of inverses to find $(f^{-1})'(6)$.

$f'(x) = 3x^2 - 1$

$(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

$(f^{-1})'(6) = \dfrac{1}{3f^{-1}(6)^2 - 1} = \dfrac{1}{3 \cdot 2^2 - 1} = \dfrac{1}{11}$

$f^{-1}(6) = ?$ is sol to $f(x) = 6$

$x^3 - x = 6$

$x = 2$

Bonus. (10pts) Let $f(x) = x^n$, where n is a positive integer, $x \geq 0$. We have justified the rule for the derivative of f using the definition by computing a limit. Use the derivative of f and either the theorem on derivatives of inverses, or implicit differentiation, to justify the rule for the derivative of $\sqrt[n]{x}$.

$f(x) = x^n$ so $f^{-1}(x) = \sqrt[n]{x}$

$(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{n(\sqrt[n]{x})^{n-1}}$

$= \dfrac{1}{n x^{\frac{n-1}{n}}} = \dfrac{1}{n} x^{-\frac{n-1}{n}}$

$= \dfrac{1}{n} x^{-1+\frac{1}{n}} = \underbrace{\dfrac{1}{n} x^{\frac{1}{n}-1}}_{\text{derivative of } x^{\frac{1}{n}}}$

Or:

$y = \sqrt[n]{x}$

$y^n = x \quad \Big| \dfrac{d}{dx}$

$n y^{n-1} y' = 1$

$y' = \dfrac{1}{n y^{n-1}} = \dfrac{1}{n(\sqrt[n]{x})^{n-1}} = \dfrac{1}{n x^{\frac{n-1}{n}}} =$ (see left)

$= \dfrac{1}{n} x^{\frac{1}{n}-1}$