Calculus 1 — Exam 2
MAT 250, Fall 2022 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\frac{d}{dx}\left(8x^5+\frac{9}{x^4}-\frac{1}{\sqrt[3]{x^2}}+\sqrt{a}\right)= 40x^4-36x^{-5}+\frac{2}{3}x^{-\frac{5}{3}}+0$

$9x^{-4}$ $x^{-\frac{2}{3}}$

$=40x^4-\frac{36}{x^5}+\frac{2}{3\sqrt[3]{x^5}}$

2. (4pts) $\frac{d}{dx}\left(x^3\cos x\right)= 3x^2\cos x+x^3(-\sin x)$

$=3x^2\cos x - x^3\sin x$

3. (6pts) $\frac{d}{ds}\frac{s^2+1}{s^2+4}= \frac{2s(s^2+4)-(s^2+1)\cdot 2s}{(s^2+4)^2}=\frac{2s^3+8s-2s^3-2s}{(s^2+4)^2}$

$=\frac{6s}{(s^2+4)^2}$

4. (6pts) $\frac{d}{d\theta}\frac{\sin^2\theta}{\cos\theta}= \frac{2\sin\theta\cos\theta\cdot\cos\theta-\sin^2\theta(-\sin\theta)}{\cos^2\theta}=\frac{2\sin\theta\cos^2\theta+\sin^3\theta}{\cos^2\theta}$

5. (6pts) $\frac{d}{dx}\sec\sqrt{x^2-3x+1}= \sec\sqrt{x^2-3x+1}\,\tan\sqrt{x^2-3x+1}\cdot\frac{1}{2\sqrt{x^2-3x+1}}\cdot(2x-3)$

$=\frac{(2x-3)\sec\sqrt{x^2-3x+1}\,\tan\sqrt{x^2-3x+1}}{2\sqrt{x^2-3x+1}}$

6. (8pts) Let $y(x)=\sin(2x)$.
a) Write the first four derivatives of y.
b) Use the pattern you found in a) to find $y^{(43)}(x)$.

$y=\sin(2x)$

$y'=\cos(2x)\cdot 2$

$y''=-\sin(2x)\cdot 2\cdot 2$

$y'''=-\cos(2x)\cdot 2\cdot 2\cdot 2$

$y^{(4)}=\sin(2x)\cdot 2\cdot 2\cdot 2\cdot 2$

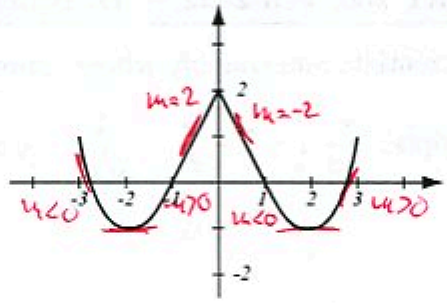
$y^{(43)}(x)=2^{43}(-\cos(2x))$

$43=40+3$

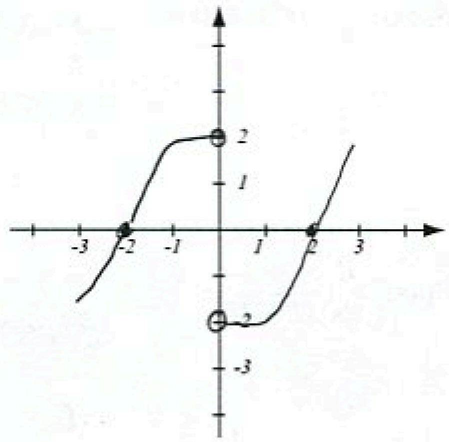
$\sin^{(40)}x=\sin x,\quad \sin^{(3)}(x)=-\cos x$

7. (10pts) The graph of the function $f(x)$ is shown at right.

a) Where is $f(x)$ not differentiable? Why?
b) Use the graph of $f(x)$ to draw an accurate graph of $f'(x)$.



a) at $x=0$ - sharp point



8. (12pts) Let $f(x) = 2x^2 - 3x$.
a) Use the limit definition of the derivative to find the derivative of the function.
b) Check your answer by taking the derivative of f using differentiation rules.
c) Write the equation of the tangent line to the curve $y = f(x)$ at point $(1, -2)$.

a) $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(h)}{h} = \lim_{h\to 0} \frac{2(x+h)^2 - 3(x+h) - (2x^2 - 3x)}{h}$

$= \lim_{h\to 0} \frac{2(x^2 + 2xh + h^2) - \cancel{3x} - 3h - \cancel{2x^2} + \cancel{3x}}{h} = \lim_{h\to 0} \frac{4xh + 2h^2 - 3h}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(4x + 2h - 3)}{\cancel{h}} = 4x - 3$

b) $f'(x) = 4x - 3$ so agrees

c) $f'(1) = 4 - 3 = 1$ $\quad y - (-2) = 1 \cdot (x - 1)$

$y = x - 1 - 2$

$y = x - 3$

9. (10pts) Let $g(x) = x^2 f(x)$ and $h(x) = f(f(x^2))$.
a) Find the general expressions for $g'(x)$ and $h'(x)$.
b) Use the table of values at right to find $g'(2)$ and $h'(2)$.

x	1	2	3	4
$f(x)$	-1	-3	3	1
$f'(x)$	-3	1	5	-1

a) $g'(x) = 2x\, f(x) + x^2 \cdot f'(x)$

$h'(x) = f'(f(x^2)) \cdot f'(x^2) \cdot 2x$

b) $g'(2) = 2 \cdot 2\, f(2) + 2^2 \cdot f'(2) = 4 \cdot (-3) + 4 \cdot 1 = -8$

$h'(2) = f'(f(4)) \cdot f'(4) \cdot 4 = f'(1) \cdot f'(4) \cdot 4 = (-3)(-1) \cdot 4 = 12$

10. (7pts) A ball thrown upwards has position given by the formula $s(t) = -5t^2 + 30t$.
a) Write the formula for the velocity of the ball at time t.
b) When does the ball reach its maximum height?
c) What is the maximal height of the ball?

a) $v(t) = s'(t) = -10t + 30$

b) At max. height when $v(t) = 0$

$-10t + 30 = 0$

$10t = 30$

$t = 3$

c) $s(3) = -5 \cdot 3^2 + 30 \cdot 3$

$= -45 + 90$

$= 45$

11. (11pts) Use implicit differentiation to find y'.

$\tan(xy) = \dfrac{x^2}{y} + y^2 \quad \Big| \dfrac{d}{dx}$

$\sec^2(xy)(1 \cdot y + xy') = \dfrac{2xy - x^2 y'}{y^2} + 2yy' \quad \Big| \cdot y^2$

$y^2 \sec^2(xy)(y + xy') = 2xy - x^2 y' + 2y^3 y'$

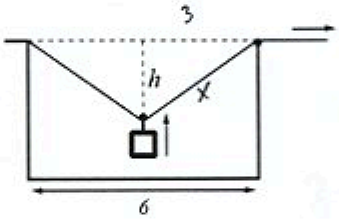
$y^3 \sec^2(xy) + xy^2 \sec^2(xy) \cdot y' = 2xy - x^2 y' + 2y^3 y'$

$y^3 \sec^2(xy) - 2xy = 2y^3 y' - x^2 y' - xy^2 \sec^2(xy)$

$y^3 \sec^2(xy) - 2xy = y'(2y^3 - x^2 - xy^2 \sec^2(xy))$

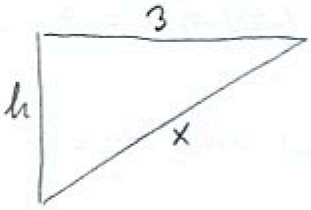
$y' = \dfrac{y^3 \sec^2(xy) - 2xy}{2y^3 - x^2 - xy^2 \sec^2(xy)}$

12. (14pts) A safe is raised using the pulley system shown, with rope pulled at right at speed 0.1 meters per second, while it is anchored at left. How fast is the distance to the top h changing when the middle pulley is 2 meters from the top? The rope shortens evenly on both sides of the safe so it stays in the center.



Know $x' = -\frac{0.1}{2}$ m/s $= -0.05$

Need h' when $h = 2$



$x^2 = h^2 + 3^2 \quad \Big| \frac{d}{dt}$

$2xx' = 2hh' + 0$

$hh' = xx'$

$h' = \frac{xx'}{h}$

$x^2 = 2^2 + 3^2$

$x^2 = 13$

$x = \sqrt{13}$

When $h = 2$, $h' = \frac{\sqrt{13}\cdot(-0.05)}{2} = -\frac{5\sqrt{13}}{200}$ m/s

Bonus. (10pts) Let $f(x) = \frac{1}{x^4}$. Use the limit definition of the derivative to find the derivative of the function and check it against the known result.

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\frac{1}{x^4} - \frac{1}{a^4}}{x-a} = \lim_{x\to a} \frac{a^4 - x^4}{x^4 a^4} \cdot \frac{1}{x-a}$$

$$= \lim_{x\to a} \frac{(a^2-x^2)(a^2+x^2)}{x^4 a^4 (x-a)} = \lim_{x\to a} \frac{\overset{-\cancel{(x-a)}}{(a-x)}(a+x)(a^2+x^2)}{x^4 a^4 \cancel{(x-a)}} = -\frac{(a+a)(a^2+a^2)}{a^4 a^4}$$

$$= -\frac{2a \cdot 2a^2}{a^8} = -\frac{4}{a^5} \qquad (x^{-4})' = -4x^{-5} = -\frac{4}{x^5}$$